Annals of Pure and Applied Logic 161 (2010) 1431–1461

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

Logic for update products and steps into the past

Joshua Sack

School of Computer Science, Reykjavík University, Iceland

ARTICLE INFO

Article history:
Received 22 October 2007
Received in revised form 14 May 2009
Accepted 7 April 2010
Available online 7 June 2010
Communicated by S.N. Artemov

MSC:
03
68

Keywords:
Completeness
Decidability
Dynamic epistemic logic
Epistemic logic
Modal logic
Temporal logic

ABSTRACT

This paper provides a sound and complete proof system for a language $\mathcal{L}_{e+Y}$ that adds to Dynamic Epistemic Logic (DEL) a discrete previous-time operator as well as single symbol formulas that partially reveal the most recent event that occurred. The completeness theorem is by filtration followed by model unraveling and other model transformations. Decidability follows from the completeness proof. The degree to which it is important to include the additional single symbol formulas is addressed in a discussion about the difficulties of the completeness for a language $\mathcal{L}_Y$ that only adds the previous-time operator to DEL. Discussion is also given regarding the completeness for a language obtained by removing common knowledge operators from $\mathcal{L}_{e+Y}$.

1. Introduction

Epistemic logic describes the knowledge and belief of agents. Adding to this an ability to describe changes in the knowledge and belief has been receiving an increasing amount of attention. Such logics can contribute to a number of fields such as artificial intelligence and computer security. One general approach to describing changes in beliefs has been to describe changes in knowledge and belief over time using Epistemic Temporal Logics (ETL) [10,7,8,13]. Another general approach, called Dynamic Epistemic Logic (DEL) [2,3,14,11,9], describes changes as a result of events that involve information being revealed to the agents in a variety of ways such as through a public or a private announcement. While ETL can express beliefs of agents in both the past and future, the causes of the belief change are given little structure. Dynamic Epistemic Logic, however, provides internal structure to the causes of the belief change, typically using relational structures called event models, and has a sophisticated method of determining new beliefs from old through such events. But DEL has very limited flexibility with regard to describing beliefs through time, and cannot reflect the past at all.

Adding temporal operators to DEL has been receiving recent attention. The paper [16] introduces a number of languages that add both past and future temporal components to DEL, and gives examples of where these languages can be used. The paper [1] focuses on the involvement of future operators, and both [12,15] have involved discrete previous-time operators. The papers [12,15,16] involve the addition of previous-time operators to variations of DEL.

This paper focuses on the proof of soundness, completeness, and decidability of a language that adds a discrete previous-time operator to DEL. As is discussed in [16], adding a previous-time operator to DEL allows us not only to express what happened in the past, but also the effect of agents being informed about past situations. Using a previous-time operator, we can also express in the logic when an agent's beliefs changed or how long it has been since something was true. In this

E-mail address: joshua.sack@gmail.com.

0168-0072/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.apal.2010.04.011

paper, time is structured similarly to the natural numbers, with a least point representing the earliest stage of time and with discrete increments. The previous-time operator also allows us to express what time it is with respect to this beginning.

Both [12,15] include proof systems involving previous-time operators that are shown to be both sound and complete with respect to semantics that have differences from the one used in this paper. The language given in [12] includes previous-time operators parametrized by full events but has no common knowledge operator, whereas this paper focuses on a language $\mathcal{L}_{e+Y}$ that has partial events as atomic formulas and a common knowledge operator. The completeness proof for the language given in [12] uses a method that builds on a completeness proof in [5] for dynamic epistemic logic constrained by a protocol. The techniques for completeness used in this paper differ from those in [12], building on the language-reduction technique of [3,9], while also employing a novel mix of model translations. Furthermore, this paper also investigates completeness for a language $\mathcal{L}_Y$ with a single (un-parametrized) previous-time operator without the company of formulas for partial events, and reveals the important role of the partial events in the completeness proof.

The paper [15] involves an un-parametrized previous-time operator without the company of partial events, and also does not include common knowledge. Although [15] places some emphasis on showing how temporal restrictions can be placed in a coherent way on the models so that they are for example ordered and non-past branching, completeness in [15] is only proved with respect to a minimally restricted class of models that do not enforce restrictions on the temporal component. This paper, on the other hand, proves completeness for structures that are guaranteed to have structural properties given by a sequence of epistemic models, with each model following the previous one according to the updating procedure used in DEL.

The primary structures used in this paper are sequences of epistemic models, which we call sequential histories. Completeness is proved by finding a model for a consistent formula, and to make it easier to represent all of the previous models, we involve for the purpose of proving completeness, a single model with a previous-time relation and suitable semantics for such a model. In this paper, we call such models with a previous-time relation *epistemic temporal models* (ETM), and we call them *epistemic temporal histories* (ETH) when they obey certain structural properties, many of which correspond to ones identified in the merging frameworks papers [5,6]. This paper then shows that every epistemic temporal history is isomorphic to some sequential history. This is similar to the developments in [15], in which an isomorphism is established between sequential histories and epistemic temporal models generated from an epistemic model by repeated applications of updates that are particular to that paper; however the isomorphism from [15] is not used for completeness. In addition to helping prove completeness, the isomorphism in this paper helps solidify the connection between the merging frameworks properties and the updates in [15].

A filtration method is used to find a finite ETM for a given consistent formula, though this ETM need not satisfy all the constraints needed to be an ETH. Thus a number of model transformations are performed. The first of these transformations is an unraveling of the model into a tree-like structure. A complete unraveling often results in an infinite model, though to establish decidability, we only unravel in the temporal direction, which can safely be trimmed to ensure the resulting model is finite. The remaining transformations involve the removal of some states and the addition of some relational connections. The truth of all formulas is not preserved through some of these transformations, but the truth of the original consistent formula will be preserved at some state, thus maintaining its satisfiability.

The organization of this paper is as follows. The next section (Section 2) describes the structures involved in DEL, and introduces the structures, called sequential histories, on which the semantics of the languages of this paper are defined. Three languages are then introduced, $\mathcal{L}_Y$, $\mathcal{L}_{Y_e}$, and $\mathcal{L}_{e+Y}$. The language $\mathcal{L}_Y$ simply adds to DEL a previous-time operator that does not depend on previous events. The language $\mathcal{L}_{Y_e}$ replaces the unconditional previous-time operator of $\mathcal{L}_Y$ with previous-time operators that depend on some aspect of the most recent event. The language $\mathcal{L}_{e+Y}$ adds to $\mathcal{L}_Y$ single symbol formulas that make assertions about some aspect of the most recent event. The languages $\mathcal{L}_{Y_e}$ and $\mathcal{L}_{e+Y}$ are equally expressive, and we choose to use $\mathcal{L}_{e+Y}$ for the proof system and completeness theorem. Section 3 introduces a proof system for $\mathcal{L}_{e+Y}$. Soundness is proved and useful provable extensions of the axioms are also given. Section 4 introduces ETMs and ETHs, which are easier to construct using a filtration than the sequential histories typically used. But the semantics of some components of the language are more difficult to define on ETMs, and hence a function is defined to translate any formula into an equivalent one for which such semantics are easier to define, using a variation of the reduction technique of both [3,9]. An isomorphism theorem showing that at ETH corresponds to a sequential history is also given. Section 5 proves completeness for $\mathcal{L}_{e+Y}$, by first defining a filtration. The filtration is a finite epistemic temporal model that need not have all the properties needed for it to correspond to a sequential history. A number of model transformations are performed to establish the desired properties. Section 6 discusses how the completeness proof for $\mathcal{L}_{e+Y}$ can be modified for languages that have common knowledge operators, and then discusses to what extent the completeness proof could be used for the language $\mathcal{L}_Y$.

2. Structures

We generally use the symbol $\mathbb{A}$ for a finite set of agents and Φ for a set of atomic propositions.

2.1. Models and update product

Definition 2.1 (*State Model*)**.** Given a set $\mathbb{A}$ of agents and a set Φ of atomic propositions, define a *state model* to be a tuple $\mathcal{S} = (S, \to_{\mathcal{S}}, \|\cdot\|_{\mathcal{S}})$, where

1. S is a set,
2. $\to_{\mathcal{S}}: \mathbb{A} \to S \times S$ is a function that assigns a binary relation $\xrightarrow{A}_{\mathcal{S}}$ over S for every agent $A \in \mathbb{A}$,
3. $\| \cdot \|_{\mathcal{S}}$ is a function mapping each proposition letter in Φ to a subset of S.

Notation 2.2. We typically drop the subscript $\mathcal{S}$ from the components of a state model when the model is understood from context.

We call each element of S a *state*. The set S is called the *carrier* set of the model $\mathcal{S}$. Define a function car that maps a state model to its carrier set. For each $A \in \mathbb{A}$, the relation $\xrightarrow{A}$ is written in infix, where $s \xrightarrow{A} t$ can be read as "in state s, A considers t possible". We call a relation with this reading an *epistemic relation*. The function $\| \cdot \|_{\mathcal{S}}$ is called a *valuation function*. We can think of the atomic propositions in Φ as being properties the valuation function assigns to the states.

A pair $(\mathcal{S}, s)$ consisting of a state model $\mathcal{S}$ and a state $s \in \mathsf{car}(\mathcal{S})$ is called a *pointed state model*. Call a sequence $H = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ of state models a *state model sequence*. We will later define a history to be a state model sequence that satisfies certain conditions, which is why we choose the letter H. We can think of the state models as representing a certain stage in time, with $\mathcal{S}_1$ being the oldest and $\mathcal{S}_n$ being the most recent. Define a function mdl that takes a state model sequence $(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ and returns the model $\mathcal{S}_n$. A pair (H, s) where $s \in \mathsf{car}(\mathsf{mdl}(H))$ is called a *pointed sequence*.

Definition 2.3 (*Splitting Function*)**.** A function that given any state model sequence H returns a subset of $\mathsf{car}(\mathsf{mdl}(H))$ is called a *splitting function*. Let $\mathfrak{S}$ be the set of all splitting functions.

We next define an event model, for which its primary difference from a state model is that the valuation function is replaced by a different function pre called a precondition function.

Definition 2.4 (*Event Model*)**.** Given a set $\mathbb{A}$ of agents, define an *event model* to be a tuple $\mathcal{E} = (E, \to_{\mathcal{E}}, \mathsf{pre}_{\mathcal{E}})$, where

1. $E = \{e_1, \ldots, e_n\}$ is a finite set, with a fixed enumeration,
2. $\to_{\mathcal{E}}: \mathbb{A} \to E \times E$ is a function assigning a binary relation $\xrightarrow{A}_{\mathcal{E}}$ to every agent $A \in \mathbb{A}$,
3. $\mathsf{pre}_{\mathcal{E}} : E \to \mathfrak{S}$ is a function assigning a splitting function to every $e \in E$.

Notation 2.5. We typically drop the subscript $\mathcal{E}$ from the components of an event model when the model is understood from context.

Each element of E is called an *event point*. We can apply the function car to event models as well as state models so that given an event model $\mathcal{E} = (E, \to, \mathsf{pre})$, $\mathsf{car}(\mathcal{E}) = E$. A pair $(\mathcal{E}, e)$ where $e \in \mathsf{car}(\mathcal{E})$ is called a *pointed event model*. For each $A \in \mathbb{A}$, the relation $\xrightarrow{A}_{\mathcal{E}}$ is written in infix, where $e \xrightarrow{A}_{\mathcal{E}} f$ can be read as "A considers f a possible occurrence if e actually happens". Due to the reading of this relation, we also call it an *epistemic relation*.

Notation 2.6. Given a state model $(S, \to_{\mathcal{S}}, \| \cdot \|_{\mathcal{S}})$ or an event model $(E, \to_{\mathcal{E}}, \| \cdot \|_{\mathcal{E}})$, if $\mathbb{B} \subseteq \mathbb{A}$ is a set of agents, we may write $\xrightarrow{\mathbb{B}}_{\mathcal{S}}$ for $\bigcup\{\xrightarrow{A}_{\mathcal{S}}: A \in \mathbb{B}\}$ and similarly $\xrightarrow{\mathbb{B}}_{\mathcal{E}}$ for $\bigcup\{\xrightarrow{A}_{\mathcal{E}}: A \in \mathbb{B}\}$. Subscripts may be dropped if the model is understood from context.

For any relation in this paper, we define the following.

Definition 2.7 (*Iterated Relations and Reflexive Transitive Closure*)**.** For a relation R over a set X, we define $R^0 = \{(x, x) : x \in X\}$ to be the smallest reflexive relation over A. For each $n \geq 0$ the *iterated relation* $R^{n+1} = RR^n$, where $RR^n = \{(x, z) :$ there exists y such that xRy and $yR^nz\}$ represents the usual composition of the relation R with R^n. We also define the *reflexive transitive closure* R^* of R to be $R^* = \bigcup_{n=0}^{\infty} R^n$.

Definition 2.8 (*Update Product*)**.** Given a state model sequence $\mathcal{H} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, where $\mathcal{S}_n = (S_n, \to_n, \| \cdot \|_n)$ and given an event model $\mathcal{E} = (E, \to_{\mathcal{E}}, \| \cdots \|_{\mathcal{E}})$, the *update product* of $\mathcal{H}$ and $\mathcal{E}$, written $\mathcal{H} \otimes \mathcal{E}$, is $(\mathcal{S}_1, \ldots, \mathcal{S}_n)$, where $\mathcal{S}_{n+1} = (S_{n+1}, \to_{n+1}, \| \cdot \|_{n+1})$ and

1. $S_{n+1} = \{(s, e) \in S_n \times E : s \in \mathsf{pre}(e)(\mathcal{S}_n)\}$,
2. $(s, e) \xrightarrow{A}_{n+1} (t, f)$ iff both $s \xrightarrow{A}_n t$ and $e \xrightarrow{A}_{\mathcal{E}} f$ (Note that $(s, e), (t, f) \in S_{n+1}$),
3. $\|p\|_{n+1} = \{(s, e) \in S_{n+1} : s \in \|p\|_n\}$.

Definition 2.9 (*Sequential History*)**.** A state model sequence $\mathcal{H} = (\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n)$ is called a *sequential history* if there is a sequence of event models $\mathcal{E}_1, \ldots, \mathcal{E}_n$, such that $\mathcal{H} = (\mathcal{S}_0) \otimes \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_n$.

We will typically call a sequential history a "history", as it will be clear what is meant until we define another structure called an "epistemic temporal history" later in this paper.

Notation 2.10. We may write $s \in \mathcal{H}$ in place of $s \in \mathsf{car}(\mathsf{mdl}(\mathcal{H}))$.

We next define more functions on histories.

Definition 2.11 (*Functions prvh, prvs, and evnt*)**.** We define function prvh (short for previous history) as follows: If $\mathcal{H} = (\mathcal{S}_1, \ldots, \mathcal{S}_n, \mathcal{S}_{n+1})$ is a history, then $\mathsf{prvh}(H) = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$. If $\mathcal{H} = (\mathcal{S})$ is a history with only one state model, then $\mathsf{prvh}(\mathcal{H}) = \emptyset$. Finally, we define the function prvh on the empty-set: $\mathsf{prvh}(\emptyset) = \emptyset$.

We define functions prvs (short for previous state) and evnt (short for event point) as extensions of projection functions π_1 and π_2, which take a pair as input and return respectively the first and second coordinates of the input. If $s = (s', e)$, then $\mathsf{prvs}(s) = s'$ and $\mathsf{evnt}(s) = e$. If $s \neq (s', e)$ for any s' or e, then $\mathsf{prvs}(s) = \mathsf{evnt}(s) = \emptyset$.

These functions are defined to be total functions on the set of histories or the set of states, which is why they are defined for initial histories and states. We define prvh on $\emptyset$, so that it can be an operator, that is so that its domain and range can be the same. This may be useful when we iterate the functions prvh and prvs. Let $\mathsf{prvh}^{n+1}(\mathcal{H}) = \mathsf{prvh}(\mathsf{prvh}^n(\mathcal{H}))$, and similarly for prvs. Defining prvh on $\emptyset$ allows us to apply prvh to itself arbitrarily many times. We know that $\mathsf{prvh}^n(\mathcal{H})$ is defined, even if we do not know whether n is greater than the length of H.

2.2. Event frame

In modal logic, the underlying relational structure of a model is called a frame, that is, the model is the frame plus the valuation function. Here, we define an event frame such that an event model is just the event frame plus a precondition function.

Definition 2.12 (*Event Frame*)**.** Given a set $\mathbb{A}$ of agents, define an *event frame* to be a tuple $\mathcal{F} = (E, \rightarrow_{\mathcal{F}})$, where

1. $E = \{e_1, \ldots, e_n\}$ is a finite set, with a fixed enumeration.
2. $\rightarrow_{\mathcal{F}}: \mathbb{A} \rightarrow E \times E$ is a function assigning a binary relation $\xrightarrow{A}_{\mathcal{F}}$ to every agent $A \in \mathbb{A}$.

Some treatments of Dynamic Epistemic Logic (DEL), such as [2], fix an event frame for a particular language. The name *signature* is sometimes used instead of the frame in order to suggest that a language depends on it.

The idea behind fixing an event frame is that in order to specify a pointed event model in a formula of the language, the formula need only capture the precondition function and a particular event point. The precondition function can be expressed using a sequence of formulas, one for every event point, and via the semantics, each formula characterizes a splitting function. As infinitely many splitting functions can be characterized using a finite alphabet of symbols, infinitely many event models can be captured using a finite alphabet of symbols. For this paper, the use of an event frame will be useful in establishing completeness.

With a fixed event frame, event models can be characterized by the precondition function alone.

Notation 2.13. If $\mathcal{F} = (E, \rightarrow)$, where E has n event points, then given a list $\vec{\varsigma} = (\varsigma_1, \ldots, \varsigma_n)$ of splitting functions, we can define a precondition function pre such that $\mathsf{pre}(e_i) = \varsigma_i$. In this way, a list of splitting functions characterizes a precondition function, which together with the event frame $\mathcal{F}$ characterizes an event model. We may write $H \otimes \vec{\varsigma}$ in place of $H \otimes \mathcal{E}$, where $\mathcal{E}$ is the event model characterized by the list $\vec{\varsigma}$ of splitting functions.

2.3. Language

The languages defined here depend on an event frame $\mathcal{F}$ and a set Φ of atomic propositions. In this section we will define three languages $\mathcal{L}_Y(\mathcal{F}, \Phi)$, $\mathcal{L}_{e+Y}(\mathcal{F}, \Phi)$ and $\mathcal{L}_{Y_e}(\mathcal{F}, \Phi)$. When $\mathcal{F}$ and Φ are understood from context, we will drop them from the notation. The language $\mathcal{L}_Y$ extends the Dynamic Epistemic Logic (DEL) in [2] with a single previous-time operator $\bar{Y}$. The language $\mathcal{L}_{e+Y}$ adds to $\mathcal{L}_Y$ formulas that express the action point of the most recent action. The language $\mathcal{L}_{Y_e}$ adds to DEL a previous-time operator Y_e that depends on the action point of the most recent action. We will see that there is a natural translation between $\mathcal{L}_{e+Y}$ and $\mathcal{L}_{Y_e}$, indicating that they are equivalent languages. The language $\mathcal{L}_{Y_e}$ will play little role outside this section, but it is defined here, for some readers may find it easier to relate to other languages they know. Later in this paper, it will be $\mathcal{L}_{e+Y}$ for which we present a complete proof system,

Definition 2.14 (*Language $\mathcal{L}_{e+Y}$, $\mathcal{L}_Y$, and $\mathcal{L}_{Y_e}$*)**.** Let Φ be a set of atomic propositions, and let $\mathcal{F} = (E, \rightarrow)$ be an event frame, where E consists of n event points. The formulas of $\mathcal{L}_{e+Y}(\mathcal{F}, \Phi)$ are given by the following Bachus–Naur form:

$$\varphi ::= \mathsf{true} \mid p \mid e \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_A\varphi \mid \Box^*_{\mathbb{B}}\varphi \mid [\psi_1, \ldots, \psi_n, e]\varphi \mid \bar{Y}\varphi$$

where $p \in \Phi$ is an atomic proposition, $e \in E$ is an event point, $A \in \mathbb{A}$ is an agent, $\mathbb{B} \subseteq \mathbb{A}$ is a set of agents, and φ, ψ, $\psi_1, \ldots, \psi_n$ are formulas.

We obtain the language $\mathcal{L}_Y(\mathcal{F}, \Phi)$ from $\mathcal{L}_{e+Y}(\mathcal{F}, \Phi)$ by removing the formula e from the Bachus–Naur form. We obtain the language $\mathcal{L}_{Y_e}(\mathcal{F}, \Phi)$ from $\mathcal{L}_Y(\mathcal{F}, \Phi)$ by adding to the Bachus–Naur form formulas of the form $Y_e\varphi$.

Note that although formulas e are eliminated from $\mathcal{L}_Y$, formulas $[\psi_1, \ldots, \psi_n, e]\varphi$ remain in the language. Thus the change is that e can no longer be a formula by itself.

Definition 2.15 (*Abbreviations*). We adopt the following abbreviations: $\mathsf{false} \equiv \neg\,\mathsf{true}$, $[\vec{\psi}e] \equiv [\psi_1, \ldots, \psi_n, e]$, $\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi \equiv \varphi \wedge \neg\psi$, and $\varphi \leftrightarrow \psi \equiv (\varphi \to \psi) \wedge (\psi \to \varphi)$. For each modality $\Box$, we define $\Box^0\varphi \equiv \varphi$ and for $n \geq 0$, $\Box^{n+1}\varphi \equiv \Box\Box^n\varphi$. We also define $\langle\vec{\psi}e\rangle\varphi \equiv \neg[\vec{\psi}e]\neg\varphi$, $\Diamond_A\varphi \equiv \neg\Box_A\neg\varphi$, $\Diamond^*_{\mathbb{B}}\varphi \equiv \neg\Box^*_{\mathbb{B}}\neg\varphi$, $\widehat{Y}\varphi \equiv \neg\bar{Y}\neg\varphi$, and $\widehat{Y}_e\varphi \equiv \neg\bar{Y}_e\neg\varphi$.

Formulas true, p, $\neg\varphi$, and $\varphi\wedge\psi$ have their usual propositional meaning, and are given a formal semantics below. Formulas of the form e, where $e \in E$, imply that the current history is not an initial history, and that the current history was created using a pointed event model, whose point is e. The symbols $\bar{Y}$, $\bar{Y}_e$, $[\vec{\psi}e]$, $\Box_A$, and $\Box^*_{\mathbb{B}}$ are called *modal operators* or *modalities*; these names will also apply to non-primitive symbols $\widehat{Y}$, $\widehat{Y}_e$, $\langle\vec{\psi}e\rangle$, $\Diamond_A$, and $\Diamond^*_{\mathbb{B}}$. Formulas of the form $\bar{Y}\varphi$ will be read as "if there were a previous stage, then φ would have been true then". Formulas of the form $\bar{Y}_e\varphi$ (which only appear in the language $\mathcal{L}_{Y_e}$) will be read as "if there were a previous stage in which e were the point of the pointed model generating the current history, then φ was true at that previous stage". Formulas of the form $\Box_A\varphi$ can be read as "A believes that φ". Formulas of the form $\Box^*_{\mathbb{B}}\varphi$ can be read as "It is common belief among the agents of $\mathbb{B}$ that φ". Formulas of the form $[\vec{\psi}e]\varphi$ are called *event modalities*, and they are read as "φ is true after any possible event characterized by $[\vec{\psi}e]$". What is meant by "possible" and "characterized by $[\vec{\psi}e]$" should become clear from the formal definition of the semantics below. The semantics is defined by a function $[\![\cdot]\!]$ from formulas to splitting functions. The splitting function tells us for what states in each model the formula is true. As we have a fixed event frame, we make use of Notation 2.13 for the update product.

Definition 2.16 (*Semantics*). The semantics for $\mathcal{L}_{Y_e}$, $\mathcal{L}_{e+Y}$ and $\mathcal{L}_Y$ is defined inductively by a function $[\![\cdot]\!]$ that takes a formula and returns a splitting function. The splitting function will take as input a history and then output the states in the most recent model of the history for which the formula is true. Let $[\![\vec{\psi}]\!] = [\![\psi_1]\!], \ldots, [\![\psi_n]\!]$.

$$
\begin{array}{lll}
s \in [\![\mathsf{true}]\!](\mathcal{H}) & \text{iff} & s \in \mathcal{H} \\
s \in [\![p]\!](\mathcal{H}) & \text{iff} & s \in \|p\|_{\mathsf{mdl}(\mathcal{H})} \\
s \in [\![e]\!](\mathcal{H}) & \text{iff} & \mathsf{evnt}(s) = e,\ \mathsf{prvh}(\mathcal{H}) \neq \emptyset, \text{ and } s \in \mathcal{H} \\
s \in [\![\neg\varphi]\!](\mathcal{H}) & \text{iff} & s \in [\![\mathsf{true}]\!](\mathcal{H}) - [\![\varphi]\!](\mathcal{H}) \\
s \in [\![\varphi \wedge \psi]\!](\mathcal{H}) & \text{iff} & s \in [\![\varphi]\!](\mathcal{H}) \cap [\![\psi]\!](\mathcal{H}) \\
s \in [\![\Box_A\varphi]\!](\mathcal{H}) & \text{iff} & t \in [\![\varphi]\!](\mathcal{H}) \text{ whenever } s \xrightarrow{A}_{\mathsf{mdl}(\mathcal{H})} t \\
s \in [\![\Box^*_{\mathbb{B}}\varphi]\!](\mathcal{H}) & \text{iff} & t \in [\![\varphi]\!](\mathcal{H}) \text{ whenever } s \xrightarrow{\mathbb{B}\ *}_{\mathsf{mdl}(\mathcal{H})} t \\
s \in [\![[\vec{\psi}e]\varphi]\!](\mathcal{H}) & \text{iff} & (s, e) \in [\![\varphi]\!](\mathcal{H} \otimes [\![\vec{\psi}]\!]) \text{ whenever } (s, e) \in \mathcal{H} \otimes [\![\vec{\psi}]\!] \\
s \in [\![\bar{Y}\varphi]\!](\mathcal{H}) & \text{iff} & \mathsf{prvs}(s) \in [\![\varphi]\!](\mathsf{prvh}(\mathcal{H})) \text{ whenever } \mathsf{prvh}(\mathcal{H}) \neq \emptyset \\
s \in [\![\bar{Y}_e\varphi]\!](\mathcal{H}) & \text{iff} & \mathsf{prvs}(s) \in [\![\varphi]\!](\mathsf{prvh}(\mathcal{H})) \text{ whenever } \mathsf{prvh}(\mathcal{H}) \neq \emptyset \text{ and } \mathsf{evnt}(s) = e.
\end{array}
$$

We can translate $\mathcal{L}_{Y_e}$ into $\mathcal{L}_{e+Y}$ by translating $Y_e\varphi$ into $e \to Y\varphi$. We can also translate $\mathcal{L}_{e+Y}$ into $\mathcal{L}_{Y_e}$ by translating e into $\neg Y_e \neg\,\mathsf{true}$ and $Y\varphi$ into $\bigwedge\{Y_e\varphi : e \in E\}$.

The primary completeness result of this paper is for $\mathcal{L}_{e+Y}$ with respect to the class of characterizable sequential histories:

Definition 2.17 (*Characterizable Sequential History*). Given a language $\mathcal{L}_{e+Y}(\mathcal{F}, \Phi)$, a sequential history $\mathcal{H} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is *characterizable*, let $\mathcal{H}_k = (\mathcal{S}_1, \ldots, \mathcal{S}_k)$ for each $k \leq n$. Then for each $k < n$, there are formulas $\vec{\psi}$ (depending on k) such that $\mathcal{H}_{k+1} = \mathcal{H}_k \otimes \vec{\psi}$.

3. Proof system

We present a proof system for $\mathcal{L}_{e+Y}$. Recall that the symbols $\to$, $\widehat{Y}$, and false are not primitive symbols, and hence most of the formulas listed below are abbreviations for the actual axioms.

3.1. Axioms and rules

We first present a set of axioms not involving knowledge.

$[\vec{\chi}e](\varphi \to \psi) \to ([\vec{\chi}e]\varphi \to [\vec{\chi}e]\psi)$	$[\vec{\chi}e]$-normality
$\bar{Y}(\varphi \to \psi) \to (\bar{Y}\varphi \to \bar{Y}\psi)$	$\bar{Y}$-normality
$[\vec{\psi}e_i]p \leftrightarrow (\psi_i \to p)$	future atomic permanence
$\bar{Y}p \leftrightarrow (\widehat{Y}\,\mathsf{true} \to p)$	past atomic permanence
$[\vec{\psi}e_i]\neg\varphi \leftrightarrow (\psi_i \to \neg[\vec{\psi}e_i]\varphi)$	event model partial functionality
$\bar{Y}\neg\varphi \leftrightarrow (\widehat{Y}\,\mathsf{true} \to \neg\bar{Y}\varphi)$	non-branching past
$[\vec{\psi}e_i]\bar{Y}\varphi \leftrightarrow (\psi_i \to \varphi)$	future past mix
$[\vec{\psi}e]e$	future event point mix
$\widehat{Y}\,\mathsf{true} \leftrightarrow \bigvee\{e : e \in E\}$	past and event point mix
$e \to \neg f$ for each $e \neq f$	uniqueness of event points

The next set of axioms involves belief, and some are expressed with conjunctions over a set of formulas. In the case that a conjunction (as in the epistemic future axiom) turns out to be over $\emptyset$ (that is, it has no conjuncts), replace it with the formula true.

$\Box_A(\varphi \to \psi) \to (\Box_A\varphi \to \Box_A\psi)$	$\Box_A$-normality
$\Box^*_{\mathbb{B}}(\varphi \to \psi) \to (\Box^*_{\mathbb{B}}\varphi \to \Box^*_{\mathbb{B}}\psi)$	$\Box^*_{\mathbb{B}}$-normality
$\Box^*_{\mathbb{B}}(\varphi \to \bigwedge\{\Box_A\varphi : A \in \mathbb{B}\}) \to (\varphi \to \Box^*_{\mathbb{B}}\varphi)$	induction
$\Box^*_{\mathbb{B}}\varphi \to \varphi \wedge \bigwedge\{\Box_A\Box^*_{\mathbb{B}}\varphi : A \in \mathbb{B}\}$	epistemic mix
$[\vec{\psi}e_i]\Box_A\varphi \leftrightarrow (\psi_i \to \bigwedge\{\Box_A[\vec{\psi}e_j]\varphi : e_i \xrightarrow{A} e_j\})$	epistemic future mix
$\bar{Y}\Box_A\varphi \to \Box_A\bar{Y}\varphi$	epistemic past mix
$\widehat{Y}\,\mathsf{true} \to \Box_A\widehat{Y}\,\mathsf{true}$	non-initial-time
$\bar{Y}\,\mathsf{false} \to \Box_A\bar{Y}\,\mathsf{false}$	initial-time
$e \to \Box_A\neg f(\text{where } \neg(e \xrightarrow{A} f))$	restriction

For the rules, we use the symbol $\vdash$ in front of a formula that is provable. We have the following standard model logic rules:

From $\vdash \varphi$ and $\vdash \varphi \to \psi$, infer $\vdash \psi$	modus ponens
From $\vdash \varphi$, infer $\vdash [\vec{\psi}e]\varphi, \vdash \Box_A\varphi, \vdash \Box^*_{\mathbb{B}}\varphi, \vdash \bar{Y}\varphi$	necessitation

We present one more rule, called the *event rule*, which will make use of sequences of event modalities, modalities of the form $[\vec{\psi}e]$. Define Ω to be the set of all sequences of one or more event modalities, and let Ω^+ add to Ω an empty string λ that has no modalities (both of these sets depend on the language). We may use any string $\alpha \in \Omega^+$ in expressing formulas. For example $\lambda\varphi$ is just φ. In order to state the event rule, we will define the following concerning Ω.

Definition 3.1 (*Syntactic Precondition Function*)**.** Define $\textsc{Pre} : \Omega^+ \to \mathcal{L}_{e+Y}$ such that

- $\textsc{Pre}(\lambda) = \mathsf{true}$,
- $\textsc{Pre}([\vec{\psi}e_i] = \psi_i$,
- $\textsc{Pre}(\alpha'\alpha) = \textsc{Pre}(\alpha') \wedge \alpha'\textsc{Pre}(\alpha)$.

Definition 3.2 (*Relations over Ω^+*)**.** For each agent $A \in \mathbb{A}$, define $\xrightarrow{A} \subseteq \Omega^+ \times \Omega^+$ to be the smallest relation for which the following hold:

- $\lambda \xrightarrow{A} \lambda$.
- $[\vec{\psi}e] \xrightarrow{A} [\vec{\chi}f]$, whenever $e \xrightarrow{A} f$ and for $1 \le k \le n$, $\psi_k = \chi_k$.
- $\alpha\alpha' \xrightarrow{A} \beta\beta'$ whenever $\alpha, \alpha', \beta, \beta' \in \Omega, \alpha \xrightarrow{A} \beta$ and $\alpha' \xrightarrow{A} \beta'$.

These two definitions do not depend on the break down of these strings. Given $\alpha \in \Omega^+$, let there be formulas χ_β for all β for which $\alpha \xrightarrow{\mathbb{B}}{}^* \beta$. Then

From $\vdash \chi_\beta \to \beta\varphi$ and $\vdash (\chi_\beta \wedge \textsc{Pre}(\beta)) \to \Box_A\chi_\gamma$ whenever $A \in \mathbb{B}, \alpha \xrightarrow{\mathbb{B}}{}^* \beta$ and $\beta \xrightarrow{A} \gamma$, infer $\vdash \chi_\alpha \to \alpha\Box^*_{\mathbb{B}}\varphi$	event rule

3.2. Soundness

Proposition 3.3. *The axioms are sound.*

Proof. The soundness proof for many of the axioms can be found for the soundness proofs of DEL. So here, I present proofs of those not easily found in other work.

Atomic permanence for past. $\bar{Y}p \leftrightarrow (\widehat{Y}\,\mathsf{true} \to p)$: The following are equivalent:

1. $s \in [\![\bar{Y}p]\!](\mathcal{H})$.
2. If $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$, then $\mathsf{prvs}(s) \in [\![p]\!](\mathsf{prvh}(\mathcal{H}))$.
3. If $s \in [\![\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$, then $\mathsf{prvs}(s) \in [\![p]\!](\mathsf{prvh}(\mathcal{H}))$.
4. If $s \in [\![\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$, then $s \in [\![p]\!](\mathcal{H})$.
5. $s \in [\![\widehat{Y}\,\mathsf{true} \to p]\!](\mathcal{H})$.

The step (3) $\Leftrightarrow$ (4) comes from the definition of valuation in the update product.

Non-branching past. $\bar{Y}\neg\varphi \leftrightarrow (\widehat{Y}\,\mathsf{true} \rightarrow \neg\bar{Y}\varphi)$: The following are equivalent:

1. $s \in [\![\bar{Y}\neg\psi]\!](\mathcal{H})$.
2. If $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$, then $\mathsf{prvs}(s) \in [\![\neg\psi]\!](\mathcal{H})$.
3. If $s \in [\![\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$, then $\mathsf{prvs}(s) \notin [\![\psi]\!](\mathcal{H})$.
4. If $s \in [\![\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$, then $s \notin [\![Y\psi]\!](\mathcal{H})$.
5. $s \in [\![\widehat{Y}\,\mathsf{true} \rightarrow \neg Y\psi]\!](\mathcal{H})$.

Future past mix. $[\vec{\psi}e_i]\bar{Y}\varphi \leftrightarrow (\psi_i \rightarrow \varphi)$: The following are equivalent:

1. $s \in [\![[\vec{\psi}e_i]\bar{Y}\varphi]\!](\mathcal{H})$.
2. If $(s, e_i) \in \mathcal{H} \otimes [\![\vec{\psi}]\!]$, then $(s, e_i) \in [\![\bar{Y}\varphi]\!](\mathcal{H} \otimes [\![\vec{\psi}]\!])$.
3. $s \in [\![\psi_i]\!](\mathcal{H})$, then $(s, e_i) \in [\![\bar{Y}\varphi]\!](\mathcal{H} \otimes [\![\vec{\psi}]\!])$.
4. If $s \in [\![\psi_i]\!](\mathcal{H})$, then $s \in [\![\varphi]\!](\mathcal{H})$.
5. $s \in [\![\psi_i \rightarrow \varphi]\!](\mathcal{H})$.

Non-initial time. $\widehat{Y}\,\mathsf{true} \rightarrow \Box_A\widehat{Y}\,\mathsf{true}$: Suppose $s \in [\![\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$. Then $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$ and $\mathsf{prvs}(s) \in [\![\,\mathsf{true}]\!](\mathsf{prvh}(\mathcal{H}))$. If $s \xrightarrow{A} s'$, then since $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$ and $\mathsf{prvs}(s') \in [\![\,\mathsf{true}]\!](\mathsf{prvh}(\mathcal{H}))$, $s' \in [\![\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$. As s' was picked arbitrarily, $s \in [\![\Box_A\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$.

Initial time. $\bar{Y}\,\mathsf{false} \rightarrow \Box_A\bar{Y}\,\mathsf{false}$: We shall argue by contrapositive. Suppose that $s \in [\![\Diamond_A\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$. Then there is an s' such that $s' \in [\![\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$. Thus $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$ (as well as $\mathsf{prvs}(s') \in [\![\,\mathsf{true}]\!](\mathsf{prvh}(\mathcal{H}))$). Hence $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$ and $\mathsf{prvs}(s) \in [\![\,\mathsf{true}]\!](\mathsf{prvh}(\mathcal{H}))$, whence $s \in [\![\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$.

Epistemic past mix. $\bar{Y}\Box_A\varphi \rightarrow \Box_A\bar{Y}\varphi$: Suppose $s \in [\![\bar{Y}\Box_A\varphi]\!](\mathcal{H})$. Then if $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$, then $\mathsf{prvs}(s) \in [\![\Box_A\varphi]\!](\mathsf{prvh}(\mathcal{H}))$. Now suppose $s \xrightarrow{A} s'$. By definition of the update product, $\mathsf{prvs}(s) \xrightarrow{A} \mathsf{prvs}(s')$ granted $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$. Now if $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$, then $\mathsf{prvs}(s') \in [\![\varphi]\!](\mathsf{prvh}(\mathcal{H}))$, and hence $s' \in [\![\bar{Y}\varphi]\!](\mathcal{H})$. As s' was picked arbitrarily, $s \in [\![\Box_A\bar{Y}\varphi]\!](\mathcal{H})$.

Restriction. $e \xrightarrow{A} \Box_A\neg f$ (where $\neg(e \xrightarrow{A} f)$): Suppose $s \in [\![e]\!](H)$, and suppose $s \xrightarrow{A} s'$. By the definition of the update product relation, $\mathsf{evnt}(s) \xrightarrow{A}_{\mathcal{F}} \mathsf{evnt}(s')$. Note that $\mathsf{evnt}(s) = e$. As it is not the case that $e \xrightarrow{A} f$, $\mathsf{evnt}(s') \neq f$. Hence $s' \notin [\![f]\!](\mathcal{H})$, and $s' \in [\![\neg f]\!](\mathcal{H})$. Hence $s \in [\![\Box_A\neg f]\!](\mathcal{H})$.

Past and event point mix. $\widehat{Y}\,\mathsf{true} \leftrightarrow \bigvee\{e : e \in E\}$: The following are equivalent:

1. $s \in [\![\widehat{Y}\,\mathsf{true}]\!](\mathcal{H})$.
2. $\mathsf{evnt}(s) = e$ for some $e \in E$, $\mathsf{prvh}(\mathcal{H}) \neq \emptyset$, and $s \in \mathcal{H}$.
3. $s \in [\![\bigvee\{e : e \in E\}]\!](\mathcal{H})$.

Uniqueness. $e \rightarrow \neg f$ for all $f \neq e$: This is immediate from the fact that evnt is a function.

Future event point mix. $[\vec{\psi}e]e$: The following are equivalent:

1. $s \in [\![[\vec{\psi}e_i]e_i]\!](\mathcal{H})$.
2. If $(s, e_i) \in \mathcal{H} \otimes [\![\vec{\psi}]\!]$, then $(s, e_i) \in [\![e_i]\!](\mathcal{H} \otimes [\![\vec{\psi}]\!])$.
3. If $s \in [\![\psi_i]\!](\mathcal{H})$, then $(s, e_i) \in [\![e_i]\!](\mathcal{H} \otimes [\![\vec{\psi}]\!])$.

The last statement is true, for $\mathsf{prvh}(\mathcal{H} \otimes [\![\vec{\psi}]\!]) = \mathcal{H} \neq \emptyset$ and $\mathsf{evnt}(s, e_i) = e_i$. $\Box$

The soundness of the rule modus ponens and generalization are similar to the standard modal and propositional arguments.

3.2.1. Soundness of the event rule

As the event rule involves strings of event models, it would be helpful to have notation that can better handle these strings.

Definition 3.4 (*Update Product Defined on Ω*). Suppose n is the number of events in the event frame $\mathcal{F}$. Define $\mathcal{H} \otimes \alpha$ for each history $\mathcal{H}$ and $\alpha \in \Omega$ as follows:

- $\mathcal{H} \otimes [\vec{\psi}e] = \mathcal{H} \otimes \vec{\psi}$.
- $\mathcal{H} \otimes (\alpha\beta) = (\mathcal{H} \otimes \alpha) \otimes \beta$ for $\alpha, \beta \in \Omega$.

Definition 3.5 (*Update Product Defined on States*). Suppose $\mathcal{S}$ is a state model, and n is the number of events in the event frame $\mathcal{F}$. We define $x \otimes \alpha$ for each $x \in S$ and $\alpha \in \Omega$ as follows:

- $s \otimes [\vec{\psi}e] = (s, e)$.
- $s \otimes (\alpha\beta) = (s \otimes \alpha) \otimes \beta$ for $\alpha, \beta \in \Omega$.

We next define a notion of equivalence on sequences of event modalities.

Definition 3.6. We write $\alpha \approx \beta$ for α and β are equivalent, where $\approx$ is the smallest relation for which the following hold.

- $[\vec{\varphi}e] \approx [\vec{\psi}f]$ whenever $e, f \in E$ and $\vec{\varphi} = \vec{\psi}$.
- $(\alpha\beta) \approx (\alpha'\beta')$ whenever $\alpha \approx \alpha'$ and $\beta \approx \beta'$.

Notice that if $\alpha \xrightarrow{A}_{\Omega} \alpha'$, then $\alpha \approx \alpha'$.

We now can extend the semantics of event modalities to strings using the following lemma.

Lemma 3.7. *Suppose $\alpha, \gamma \in \Omega$, and $\alpha \approx \alpha'$. Then the following are equivalent.*

- $s \in [\![\alpha\varphi]\!](\mathcal{H})$.
- $(s \otimes \alpha) \in [\![\varphi]\!](\mathcal{H} \otimes \alpha')$ *whenever* $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$.

Proof. We prove this by induction on the structure of α. The base case $\alpha = [\vec{\psi}e]$ follows from definitions; note that as $\alpha \approx \alpha'$, it is the case that $\alpha' = [\vec{\psi}e']$ for some $e' \in E$. Suppose for α and β, if $\alpha' \approx \alpha$ and $\beta' \approx \beta$, then for every pointed history $\mathcal{H}$ and every φ, the statement of the lemma holds (in the case of the β, replace every α by β and every α' be β'). Then the following are equivalent.

1. $s \in [\![\neg\alpha\beta\varphi]\!](\mathcal{H})$.
2. $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$ and $(s \otimes \alpha) \in [\![\neg\beta\varphi]\!](\mathcal{H} \otimes \alpha')$.
3. $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$, $(s \otimes \alpha) \in [\![\textsc{Pre}(\beta)]\!](\mathcal{H} \otimes \alpha')$, and $((s \otimes \alpha) \otimes \beta) \in [\![\neg\varphi]\!]((\mathcal{H} \otimes \alpha) \otimes \beta')$.
4. $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$, $s \in [\![\alpha\textsc{Pre}(\beta)]\!](\mathcal{H})$, and $((s \otimes \alpha) \otimes \beta) \in [\![\neg\varphi]\!](\mathcal{H} \otimes \alpha') \otimes \beta')$.
5. $s \in [\![\textsc{Pre}(\alpha\beta)]\!](\mathcal{H})$ and $(s \otimes (\alpha\beta)) \in [\![\neg\varphi]\!](\mathcal{H} \otimes (\alpha'\beta'))$.

The step (1) $\Leftrightarrow$ (2) uses the inductive hypothesis for α. The step (2) $\Leftrightarrow$ (3) uses the inductive hypothesis for β. The step (3) $\Leftrightarrow$ (4) uses the inductive hypothesis for α. The step (4) $\Leftrightarrow$ (5) is by the definitions. Finally notice that $\alpha\beta \approx \alpha'\beta'$. $\square$

It is also helpful to establish a clear relationship between the precondition function Pre and membership in the model.

Lemma 3.8. *Suppose $\alpha \in \Omega$. Then $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$ iff $(s \otimes \alpha) \in (\mathcal{H} \otimes \alpha)$.*

Proof. Let us prove this simultaneously by structural induction on α. The base case $\alpha = [\vec{\psi}e]$ follows from definitions.

Assume the desired result for α and β for all pointed histories and formulas φ. Then the following are equivalent.

1. $s \in [\![\textsc{Pre}(\alpha\beta)]\!](\mathcal{H})$.
2. $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$ and $s \in [\![\alpha\textsc{Pre}(\beta)]\!](\mathcal{H})$.
3. $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$ and $(s \otimes \alpha) \in [\![\textsc{Pre}(\beta)]\!](\mathcal{H} \otimes \alpha)$.
4. $((s \otimes \alpha) \otimes \beta) \in ((H \otimes \alpha) \otimes \beta)$.
5. $(s \otimes (\alpha\beta)) \in (H \otimes (\alpha\beta))$.

The step (1) $\Leftrightarrow$ (2) comes from the definition of Pre and the semantics. The step (2) $\Leftrightarrow$ (3) comes from Lemma 3.7 applied to α. The step (3) $\Leftrightarrow$ (4) comes from the inductive hypothesis for β, as well as the fact that $(s \otimes \alpha) \in [\![\textsc{Pre}(\beta)]\!](\mathcal{H} \otimes \alpha)$ implies $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$. $\square$

The following lemma extends the behavior of the relation in the update product to handle sequences of update products induced by sequences of event modalities.

Lemma 3.9. *Suppose $\alpha \in \Omega$. Then the following are equivalent.*

(a) $t \xrightarrow{A}_{\mathcal{H}\otimes\alpha} t'$.

(b) *There exists $s, s' \in \mathcal{H}$ and $\beta, \beta' \in \Omega$ such that $\beta \approx \alpha$, $s \in [\![\textsc{Pre}(\beta)]\!](\mathcal{H})$, $s' \in [\![\textsc{Pre}(\beta')]\!](\mathcal{H})$, $s \xrightarrow{A}_{\mathcal{H}} s'$, and $\beta \xrightarrow{A}_{\Omega} \beta'$.*

Proof. We prove this by induction on the length of α. The base case where $\alpha = [\vec{\psi}e]$ follows directly from the definitions.

Now suppose that $\alpha = \gamma\epsilon$, and suppose the desired result holds for γ and for ϵ for any history $\mathcal{H}$. The following are equivalent:

1. $t \xrightarrow{A}_{\mathcal{H}\otimes\alpha} t'$.
2. There exist $u, u' \in \mathcal{H} \otimes \gamma$ and $\zeta, \zeta' \in \Omega$, such that $\epsilon \approx \zeta$, $u \in [\![\textsc{Pre}(\zeta)]\!](\mathcal{H} \otimes \gamma)$, $u' \in [\![\textsc{Pre}(\zeta')]\!](\mathcal{H} \otimes \gamma)$, $u \xrightarrow{A}_{\mathcal{H}\otimes\gamma} u'$, and $\zeta \xrightarrow{A}_{\Omega} \zeta'$.
3. There exist $s, s' \in \mathcal{H}$ and $\delta, \delta', \zeta, \zeta' \in \Omega$, such that $\gamma \approx \delta$, $\epsilon \approx \zeta$, $s \in [\![\textsc{Pre}(\delta)]\!](\mathcal{H})$, $s' \in [\![\textsc{Pre}(\delta')]\!](\mathcal{H})$, $(s \otimes \delta) \in [\![\textsc{Pre}(\zeta)]\!](\mathcal{H} \otimes \gamma)$, $(s' \otimes \delta') \in [\![\textsc{Pre}(\zeta')]\!](\mathcal{H} \otimes \gamma)$, $s \xrightarrow{A}_{\mathcal{H}} s'$, $\delta \xrightarrow{A}_{\Omega} \delta'$, and $\zeta \xrightarrow{A}_{\Omega} \zeta'$.
4. There exist $s, s' \in \mathcal{H}$ and $\delta, \delta', \zeta, \zeta' \in \Omega$, such that $\gamma \approx \delta$, $\epsilon \approx \zeta$, $s \in [\![\textsc{Pre}(\delta)]\!](\mathcal{H})$, $s' \in [\![\textsc{Pre}(\delta')]\!](\mathcal{H})$, $s \in [\![\delta\textsc{Pre}(\zeta)]\!](\mathcal{H})$, $s' \in [\![\delta'\textsc{Pre}(\zeta')]\!](\mathcal{H})$, $s \xrightarrow{A}_{\mathcal{H}} s'$, $\delta \xrightarrow{A}_{\Omega} \delta'$, and $\zeta \xrightarrow{A}_{\Omega} \zeta'$.
5. There exist $s, s' \in \mathcal{H}$, $\beta, \beta' \in \Omega$, such that $\alpha \approx \beta$, $s \in [\![\textsc{Pre}(\beta)]\!](\mathcal{H})$, $s' \in [\![\textsc{Pre}(\beta')]\!](\mathcal{H})$, $s \xrightarrow{A}_{\mathcal{H}} s'$, and $\beta \xrightarrow{A}_{\Omega} \beta'$.

The step (1) $\Leftrightarrow$ (2) is by the inductive hypothesis, letting $t = u \otimes \zeta$ and $t' = u \otimes \zeta'$. For the step (2) $\Leftrightarrow$ (3), we use the inductive hypothesis, letting $u = s \otimes \delta$ and $u' = s \otimes \delta'$. The step (3) $\Leftrightarrow$ (4) follows from Lemma 3.7. For the step (4) $\Leftrightarrow$ (5), we let $\beta = \delta\zeta$ and $\beta' = \delta'\zeta'$, and recall that $\textsc{Pre}(\delta\zeta) = \textsc{Pre}(\delta) \wedge \delta\textsc{Pre}(\zeta)$. $\square$

The following lemma provides us with a necessary and sufficient condition for the existence of a path that will form the foundation of the proof of the event rule's soundness. This lemma will not only be used for the soundness of the event rule, but also for the proof of Lemma 4.17.

Lemma 3.10. *$s \in [\![\neg\alpha\Box^*_{\mathbb{B}}\varphi]\!](\mathcal{H})$ iff there is a sequence of states from $\mathcal{H}$*

$$s = s_0 \xrightarrow{A_1} s_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} s_{k-1} \xrightarrow{A_k} s_k$$

where $k \geq 0$ and each $A_i \in \mathcal{B}$, and also a sequence in Ω^+

$$\alpha = \alpha_0 \xrightarrow{A_1} \alpha_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} \alpha_{k-1} \xrightarrow{A_k} \alpha_k$$

such that $s_i \in [\![\textsc{Pre}(\alpha_i)]\!](\mathcal{H})$ for all i, where $0 \leq i < k$, and $s_k \in [\![\neg\alpha_k\varphi]\!](\mathcal{H})$.

Proof. The proof here is mostly from [3]. The following are equivalent.

1. $s \in [\![\neg\alpha\Box^*_{\mathbb{B}}\varphi]\!](\mathcal{H})$.
2. $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$ and $(s \otimes \alpha) \in [\![\neg\Box^*_{\mathbb{B}}\varphi]\!](\mathcal{H} \otimes \alpha)$.
3. $s \in [\![\textsc{Pre}(\alpha)]\!](\mathcal{H})$ and for some $k \geq 0$, there exists a sequence in $\mathcal{H} \otimes \alpha$

$$(s \otimes \alpha) = t_0 \xrightarrow{A_1} t_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} t_{k-1} \xrightarrow{A_k} t_k$$

   such that $A_i \in \mathbb{B}$ and $t_k \in [\![\neg\varphi]\!](\mathcal{H} \otimes \alpha)$.
4. There are sequences of $s = s_0, \ldots, s_k$ and $\alpha_0, \ldots, \alpha_k$ as in the statement of this lemma.

For (1) $\Leftrightarrow$ (2), use Lemma 3.7. The step (2) $\Leftrightarrow$ (3) comes from the definition of the semantics of $\neg\Box^*_{\mathbb{B}}\varphi$. For (3) $\Leftrightarrow$ (4), iterate Lemma 3.9 and use Lemma 3.7 for step k. $\square$

Proposition 3.11. *The event rule is sound.*

Proof. If $\alpha = \lambda$ or $\alpha = [\vec{\psi}e]$, then the proof of soundness in [9] (Proposition 6.37 in [9]) can be applied here. For longer strings, we follow the proof in [3], which goes as follows. Suppose $\alpha \in \Omega$ and for every β such that $\alpha \xrightarrow{\mathbb{B}}{}^* \beta$ there are formulas χ_β such that whenever $\alpha \xrightarrow{\mathbb{B}} \beta$ and $\beta \xrightarrow{A} \gamma$ for some $A \in \mathbb{A}$, the following event rule validity assumptions hold:

(a) $[\![\chi_\beta \rightarrow \beta\psi]\!] = [\![\mathsf{true}]\!]$
(b) $[\![(\chi_\beta \wedge \textsc{Pre}(\beta)) \rightarrow \Box_A\chi_\gamma]\!] = [\![\mathsf{true}]\!]$.

We wish to show that $[\![\chi_\alpha \rightarrow \alpha\Box^*_{\mathbb{B}}\psi]\!] = [\![\mathsf{true}]\!]$. Suppose that $s \in [\![\chi_\alpha]\!](\mathcal{H})$, and for a contradiction that $s \in [\![\neg\alpha\Box^*_{\mathbb{B}}\psi]\!](\mathcal{H})$. By Lemma 3.10, we have a sequence of states in $\mathcal{H}$

$$s = s_0 \xrightarrow{A_1} s_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} s_{k-1} \xrightarrow{A_k} s_k$$

where $k \geq 0$ and each $A_i \in \mathbb{B}$, and also a sequence it Ω

$$\alpha = \alpha_0 \xrightarrow{A_1} \alpha_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} \alpha_{k-1} \xrightarrow{A_k} \alpha_k$$

such that $s_i \in [\![\textsc{Pre}(\alpha_i)]\!](\mathcal{H})$ for all i, where $0 \leq i < k$, and $s_k \in [\![\neg\alpha_k\psi]\!](\mathcal{H})$. First, if $k = 0$, we would have that $s \in [\![\neg\alpha\psi]\!](\mathcal{H})$. By the event rule validity assumption (a), $[\![\chi_\alpha \rightarrow \alpha\psi]\!] = [\![\mathsf{true}]\!]$, whence $s \in [\![\chi_\alpha \rightarrow \alpha\psi]\!](\mathcal{H})$. By our initial assumption that $s \in [\![\chi_\alpha]\!](\mathcal{H})$, we get $s \in [\![\alpha\psi]\!](\mathcal{H})$, contradicting the result of the lemma that $s_k \in [\![\neg\alpha_k\psi]\!](\mathcal{H})$.

If $k > 0$, we show by induction on $1 \leq i \leq k$ that $s_i \in [\![\chi_{\alpha_i}]\!](\mathcal{H})$. The case where $i = 0$ comes from the initial assumption that $s \in [\![\chi_\alpha]\!](\mathcal{H})$. Assume that $s_i \in [\![\chi_{\alpha_i}]\!](\mathcal{H})$. By Lemma 3.10, $s_i \in [\![\textsc{Pre}(\alpha_i)]\!](\mathcal{H})$. From this and the event rule validity assumption (b), $s_i \in [\![\Box_{A_{i+1}}\chi_{\alpha_{i+1}}]\!](\mathcal{H})$. Hence $s_{i+1} \in [\![\chi_{\alpha_{i+1}}]\!](\mathcal{H})$. This completes our induction.

In particular, $s_k \in [\![\chi_{\alpha_k}]\!](\mathcal{H})$. Using again the event rule validity assumption (a), we have $s_k \in [\![\alpha_k\psi]\!](\mathcal{H})$. $\square$

3.3. Provable equivalence and examples of provable formulas

Definition 3.12 (*Provable Equivalence*)**.** We say that sentences φ and ψ are provably equivalent, and write $\varphi \equiv \psi$, iff $\vdash \varphi \leftrightarrow \psi$.

Proposition 3.13 (*Extended Event Model Partial Functionality*)**.** $\vdash \alpha\neg\varphi \leftrightarrow (\textsc{Pre}(\alpha) \to \neg\alpha\varphi)$, *for every* $\alpha \in \Omega$.

Proof. We prove this by induction on α. The base case for basic programs comes precisely from the axiom *event model partial functionality*.

Suppose the result holds for α. For the inductive step for basic actions, we see that the following are provably equivalent.

1. $[\vec{\psi} e_i]\alpha\neg\varphi$.
2. $[\vec{\psi} e_i](\textsc{Pre}(\alpha) \to \neg\alpha\varphi)$.
3. $[\vec{\psi} e_i]\neg(\textsc{Pre}(\alpha) \wedge \alpha\varphi)$.
4. $\psi_i \to \neg[\vec{\psi} e_i](\textsc{Pre}(\alpha) \wedge \alpha\varphi)$.
5. $\psi_i \to \neg([\vec{\psi} e_i]\textsc{Pre}(\alpha) \wedge [\vec{\psi} e_i]\alpha\varphi)$.
6. $(\psi_i \to \neg[\vec{\psi} e_i]\textsc{Pre}(\alpha)) \vee \neg[\vec{\psi} e_i]\alpha\varphi$.
7. $[\vec{\psi} e_i]\neg\textsc{Pre}(\alpha) \vee \neg[\vec{\psi}_i e_i]\alpha\varphi$.
8. $\langle\vec{\psi} e_i\rangle\textsc{Pre}(\alpha) \to \neg[\vec{\psi}_i e_i]\alpha\varphi$.
9. $\textsc{Pre}([\vec{\psi} e_i]\alpha) \to \neg[\vec{\psi}_i e_i]\alpha\varphi$.

The step $\vdash (2) \leftrightarrow (3)$ uses the inductive hypothesis. $\Box$

Proposition 3.14 (*Extended Epistemic Future Mix*)**.** $\vdash \alpha\Box_A\varphi \leftrightarrow (\textsc{Pre}(\alpha) \to \bigwedge\{\Box_A\beta\varphi : \alpha \xrightarrow{A}_{\Omega} \beta\})$.

Proof. This proof is mostly from [3] (known as action-knowledge). It is proved by induction on α.

If α is of the form $[\vec{\psi} e_i]$, then we simply have the axiom *epistemic future mix* in the form we know it.

So assume the desired result for α' and α; we prove it for $\alpha'\alpha$. We will show that

$$\vdash \alpha'\alpha\Box_A\varphi \leftrightarrow \left(\textsc{Pre}(\alpha'\alpha) \to \bigwedge\{\Box_A\beta'\beta\varphi : (\alpha'\alpha) \xrightarrow{A} (\beta'\beta)\}\right). \tag{1}$$

We start by using the inductive hypothesis on α to get the equivalence $\vdash \alpha\Box_A\varphi \leftrightarrow (\textsc{Pre}(\alpha) \to \bigwedge\{\Box_A\beta\varphi : \alpha \xrightarrow{A} \beta\})$. We then use necessitation and normality multiple times, and get

$$\vdash \alpha'\alpha\Box_A\varphi \leftrightarrow \left(\alpha'\textsc{Pre}(\alpha) \to \bigwedge\{\alpha'\Box_A\beta\varphi : \alpha \xrightarrow{A} \beta\}\right). \tag{2}$$

By the inductive hypothesis on α', we have that for all β

$$\vdash \alpha'\Box_A\beta\varphi \leftrightarrow \left(\textsc{Pre}(\alpha') \to \bigwedge\{\Box_A\beta'\beta\varphi ; \alpha' \xrightarrow{A} \beta'\}\right).$$

This and (2) leads the provable equivalence of $\alpha'\alpha\Box_A\varphi$ and

$$\alpha'\textsc{Pre}(\alpha) \to \left(\textsc{Pre}(\alpha') \to \bigwedge\{\Box_A\beta'\beta\varphi : \alpha \xrightarrow{A}_{\beta}, \alpha' \xrightarrow{A} \beta'\}\right).$$

By definition, $\textsc{Pre}(\alpha'\alpha) = \textsc{Pre}(\alpha') \wedge \alpha'\textsc{Pre}(\alpha)$. In addition, we have by definition $\alpha'\alpha \xrightarrow{A} \beta'\beta$ iff $\alpha' \xrightarrow{A} \beta'$ and $\alpha \xrightarrow{A} \beta$. Using these observations and some propositional reasoning, we get (1), as desired. $\Box$

Proposition 3.15. $\vdash \alpha\Box^*_{\mathbb{B}}\varphi \leftrightarrow (\textsc{Pre}(\alpha) \to \bigwedge\{\Box_A\beta\Box^*_{\mathbb{B}}\varphi : \alpha \xrightarrow{A}_{\Omega} \beta\})$.

Proof. The following are provably equivalent.

1. $\alpha\Box^*_{\mathbb{B}}\varphi$.
2. $\alpha\Box_A\Box^*_{\mathbb{B}}\varphi$.
3. $\textsc{Pre}(\alpha) \to \bigwedge\{\Box_A\beta\Box^*_{\mathbb{B}}\varphi : \alpha \xrightarrow{A}_{\Omega} \beta\}$.

The first provable equivalence is from the axiom *epistemic mix*, the many applications of necessitation and modal logic. The second provable equivalence is from Proposition 3.14 (extended epistemic future mix). $\Box$

Proposition 3.16. $\vdash \widehat{Y}\Box_A\varphi \to \Box_A\widehat{Y}\varphi$.

Proof. Each of these steps holds.

1. $\vdash \widehat{Y}\Box_A\varphi \to (\bar{Y}\Box_A\varphi \wedge \widehat{Y}\,\text{true})$ by *non-branching past*.
2. $(\bar{Y}\Box_A\varphi \wedge \widehat{Y}\,\text{true}) \to (\Box_A\bar{Y}\varphi \wedge \widehat{Y}\,\text{true})$ by *epistemic past mix* and propositional logic.
3. $(\Box_A\bar{Y}\varphi \wedge \widehat{Y}\,\text{true}) \to (\Box_A\bar{Y}\varphi \wedge \Box_A\widehat{Y}\,\text{true})$ by *non-initial time* and propositional logic.
4. $(\Box_A\bar{Y}\varphi \wedge \Box_A\widehat{Y}\,\text{true}) \to \Box_A\widehat{Y}\varphi$. $\Box$

The following proposition helps characterize the fact that every epistemically related state must be the "same time" as the original.

Proposition 3.17. *The following hold for each n.*

(a) $\vdash \bar{Y}^n\, \textsf{false} \to \Box_A \bar{Y}^n\, \textsf{false}$ *(extended initial-time).*
(b) $\vdash \widehat{Y}^n\, \textsf{true} \to \Box_A \widehat{Y}^n\, \textsf{true}$ *(extended non-initial-time).*

Proof. We prove both by induction on n. The base cases for (a) and (b) are the axioms $\bar{Y}\,\textsf{false} \to \Box_A \bar{Y}\,\textsf{false}$ and $\widehat{Y}\,\textsf{true} \to \Box_A \widehat{Y}\,\textsf{true}$ respectively. For the inductive step for (a), assume $\vdash \bar{Y}^n\,\textsf{false} \to \Box_A \bar{Y}\,\textsf{false}$. Then $\vdash \bar{Y}\bar{Y}^n\,\textsf{false} \to \bar{Y}\Box_A \bar{Y}^n\,\textsf{false}$ and our desired result comes from the repeated applications of the axiom $\vdash \bar{Y}\Box_A\varphi \to \Box_A \bar{Y}\varphi$ and *modus ponens*. The inductive step for (b) is similar, but uses Proposition 3.16 instead of *epistemic past mix*. □

Proposition 3.18. *If $i \neq j$, then*

$$\vdash [\vec{\psi}e_i]e_j \leftrightarrow \neg\psi_i.$$

Proof. The following are provably equivalent:

1. $[\vec{\psi}e_i]e_j$
2. $[\vec{\psi}e_i]e_j \wedge \textsf{true}$
3. $[\vec{\psi}e_i]e_j \wedge [\vec{\psi}e_i]e_i$
4. $[\vec{\psi}e_i](e_j \wedge e_i)$
5. $[\vec{\psi}e_i]\neg\,\textsf{true}$
6. $\psi_i \to \neg[\vec{\psi}e_i]\,\textsf{true}$
7. $\neg\psi_i$. □

4. Epistemic temporal models

Capturing a sequence of state models from maximal consistent sets can be challenging, and it may be easiest to first consider one model whose states constitute the union of the state models we wish to have in our history. We thus define an *Epistemic Temporal Model* (ETM), that augments a state model with two components, one of which is a previous-time relation Y and the other being a function ε assigning the event point that is true at a given state.

Definition 4.1 (*Epistemic Temporal Model*)**.** Given an event frame $\mathcal{F}$ (with $\mathbb{A}$ the corresponding set of agents and E the set of event points) and a set Φ of atomic propositions, define an *epistemic temporal model* (ETM) to be a tuple $\mathcal{M} = (S, \to, \|\cdot\|, Y, \varepsilon)$, where

1. S is a set,
2. $\to: \mathbb{A} \to S \times S$ is a function that assigns a binary relation $\xrightarrow{A}$ over S for every agent $A \in \mathbb{A}$,
3. $\|\cdot\|$ is a function mapping each proposition letter in Φ to a subset of S,
4. $Y \subseteq S \times S$ is a binary relation over S,
5. $\varepsilon : S \to E \cup \{\emptyset\}$ is a function assigning either an event point or $\emptyset$ to every state in S.

Ideally the relation Y will partition an ETM into state models so that Y relates a state in what shall be one state model to a state in what shall be the previous state model. While the semantics for sequence histories is dynamic, because a new history is constructed every time past or future operators are considered, we aim to define a static semantics for ETMs, that is a semantics involving just the one model. Our aim will be for no new instances of models to be considered in the description of the semantics, just the one original model. The involvement of the relation Y in an ETM will make the semantics of the past easy: the operator $\bar{Y}$ will be defined in terms of the relation Y just as $\Box_A$ is defined in terms of the epistemic relation $\xrightarrow{A}$. But the future still presents a challenge, and we will find it helpful to reduce in the language the occurrences of event operators, operators of the form $[\vec{\psi}e]$. We will do this by defining a sublanguage of $\mathcal{L}_{e+Y}$, that has the reduced occurrences of event modalities, and then translating every $\mathcal{L}_{e+Y}$ formula into a provably equivalent in the sublanguage.

4.1. Translation

To simplify the semantics for event operators in the ETM semantics, we limit the occurrences of event modalities so that strings of event modalities only occur before common knowledge modalities $\Box^*_{\mathbb{B}}$. More specifically, we define a sublanguage of $\mathcal{L}_{e+Y}$, called $\mathcal{L}_{\textsf{rf}}$ (where rf stands for restricted form).

Definition 4.2 (*Language $\mathcal{L}_{rf}$*). Given an event frame $\mathcal{F} = (E, \rightarrow_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ and a set Φ of proposition letters, let $\mathcal{L}_{rf}(\mathcal{F}, \Phi)$ be defined to be the set of formulas in the following two sorted system with *formulas* and *event formulas*. Formulas are given by the Bachus–Naur form:

$$\varphi ::= \text{true} \mid p \mid e \mid \neg\varphi \mid \varphi \wedge \psi \mid \eta \mid \Box_A\varphi \mid \Box^*_{\mathbb{B}}\varphi \mid \bar{Y}\varphi$$

where $p \in \Phi$ is an atomic proposition, $e \in E$ is an event point, $A \in \mathbb{A}$ is an agent, $\mathbb{B} \subseteq \mathbb{A}$ is a set of agents, and φ, ψ, $\psi_1, \ldots, \psi_n$ are formulas, and η is an event formula.

Event formulas are given by

$$\eta ::= [\psi_1, \ldots, \psi_n, e]\Box^*_{\mathbb{B}}\varphi \mid [\psi_1, \ldots, \psi_n, e]\eta$$

where φ, ψ, $\psi_1, \ldots, \psi_n$ are formulas and η is an event formula.

The following definition extends one in [9]. In what follows, let $\{c(\vec{\psi})\} = \{c(\psi_1), \ldots, c(\psi_n)\}$.

Definition 4.3 (*Complexity*). The complexity of a formula in $\mathcal{L}_{e+Y}(\mathcal{F}, \Phi)$ is given by a function $c : \mathcal{L}_{e+Y} \rightarrow \mathbb{N}$, defined inductively by

$$\begin{array}{lcl|lcl}
c(\text{true}) & = & 1 & c(\Box_A\varphi) & = & 1 + c(\varphi) \\
c(p) & = & 1 & c(\Box^*_{\mathbb{B}}\varphi) & = & 1 + c(\varphi) \\
c(e) & = & 1 & c(\bar{Y}\varphi) & = & 1 + c(\varphi) \\
c(\neg\varphi) & = & 1 + c(\varphi) & c([\vec{\psi}e]\varphi) & = & (4 + |E| + \max\{c(\vec{\psi})\})c(\varphi) \\
c(\varphi \wedge \psi) & = & 1 + \max\{c(\varphi), c(\psi)\} & & &
\end{array}$$

where $|E|$ is the size of the fixed set E of event points.

In what follows, let $[t(\vec{\psi}), e] = [t(\psi_1), \ldots, t(\psi_n), e]$. Let α represent a string of 0 or more possibly but not necessarily distinct event modalities.

Definition 4.4 (*Translation*). Define the following translation:

$$\begin{array}{lcl|lcl}
t(\text{true}) & = & \text{true} & t(\alpha[\vec{\psi}e]\,\text{true}) & = & t(\alpha\,\text{true}) \\
t(p) & = & p & t(\alpha[\vec{\psi}e_i]p) & = & t(\alpha(\psi_i \rightarrow p)) \\
t(e) & = & e & t(\alpha[\vec{\psi}e_i]e_i) & = & t(\alpha\,\text{true}) \\
 & & & t(\alpha[\vec{\psi}e_i]e_k) & = & t(\alpha\neg\psi_i) \quad (\text{where } i \neq k) \\
t(\neg\varphi) & = & \neg t(\varphi) & t(\alpha[\vec{\psi}e_i]\neg\varphi) & = & t(\alpha(\psi_i \rightarrow \neg[\vec{\psi}e_i]\varphi)) \\
t(\varphi_1 \wedge \varphi_2) & = & t(\varphi_1) \wedge t(\varphi_2) & t(\alpha[\vec{\psi}e](\varphi_1 \wedge \varphi_2)) & = & t(\alpha([\vec{\psi}e]\varphi_1 \wedge [\vec{\psi}e]\varphi_2)) \\
t(\bar{Y}\varphi) & = & \bar{Y}t(\varphi) & t(\alpha[\vec{\psi}e_i]\bar{Y}\varphi) & = & t(\alpha(\psi_i \rightarrow \varphi)) \\
t(\Box_A\varphi) & = & \Box_A t(\varphi) & t(\alpha[\vec{\psi}e_i]\Box_A\varphi) & = & t\left(\alpha\left(\psi_i \rightarrow \bigwedge_{\{e_k : e_i \xrightarrow{A} e_k\}} \Box_A[\vec{\psi}e_k]\varphi\right)\right) \\
t(\Box^*_{\mathbb{B}}\varphi) & = & \Box^*_{\mathbb{B}}t(\varphi) & t([\vec{\psi}e]\alpha\Box^*_{\mathbb{B}}\varphi) & = & [t(\vec{\psi})e]t(\alpha\Box^*_{\mathbb{B}}\varphi)
\end{array}$$

Definition 4.5 (*Subformula Function*). For a formula φ, let $\mathsf{sub}(\varphi)$ be the set of all subformulas of φ.

To show this is well defined, we observe the following proposition.

Proposition 4.6. *The following hold:*

1. $c(\varphi) \geq c(\psi)$ *for all* $\psi \in sub(\varphi)$
2. $c(\alpha\varphi) > c(\alpha\psi)$ *whenever* $c(\varphi) > c(\psi)$
3. $c([\vec{\psi}e_i]p) > c(\psi_i \rightarrow p)$
4. $c([\vec{\psi}e_i]e_i) > c(\mathit{true})$
5. $c([\vec{\psi}e_i]e_k) > c(\neg\psi_i)$ *(where* $i \neq k$*)*
6. $c([\vec{\psi}e_i]\neg\varphi) > c(\psi_i \rightarrow \neg[\vec{\psi}e_i]\varphi)$
7. $c([\vec{\psi}e](\varphi_1 \wedge \varphi_2)) > c([\vec{\psi}e]\varphi_1 \wedge [\vec{\psi}e]\varphi_2)$
8. $c([\vec{\psi}e_i]\bar{Y}\varphi) > c(\psi_i \rightarrow \varphi)$
9. $c([\vec{\psi}e_i]\Box_A\varphi) > c\left(\psi_i \rightarrow \bigwedge\{\Box_A[\vec{\psi}e_k]\varphi : e_i \xrightarrow{A}_{\mathcal{F}} e_k\}\right)$.

Proof. Most of these cases are from the book [9]. So only select cases are proved here. For (8):

$$\begin{aligned}
c([\vec{\psi}e_i]\bar{Y}\varphi) &= (4 + |E| + \max\{c(\vec{\psi})\})(1 + c(\varphi)) \\
&> 2 + \max\{2 + c(\psi_i), 1 + c(\varphi)\} \\
&= c(\psi_i \rightarrow \varphi).
\end{aligned}$$

For (9):

$$
\begin{aligned}
c([\vec{\psi}e_i]\Box_A\varphi) &= (4+|E|+\max\{c(\vec{\psi})\})(1+c(\varphi)) \\
&> 4+|E|+(4+|E|+\max\{c(\vec{\psi})\})c(\varphi) \\
&= 2+\max\{2+c(\psi_i),\,1+|E|+\max\{1+(4+|E|+\max\{c(\vec{\psi})\})c(\varphi)\}\} \\
&\geq c(\neg(\neg\neg\psi_i \wedge \neg \bigwedge\{\Box_A[\vec{\psi}e_k]\varphi : e_i \xrightarrow{A} e_k\})) \\
&= c(\psi_i \to \bigwedge\{\Box_A[\vec{\psi}e_k]\varphi : e_i \xrightarrow{A}_{\mathcal{F}} e_k\}). \quad \Box
\end{aligned}
$$

We now wish to show that the transformation t behaves in a desirable fashion. One can see that for all formulas φ in $\mathcal{L}_{e+Y}$, $t(\varphi)$ is in $\mathcal{L}_{\mathrm{rf}}$, since any formula not in $\mathcal{L}_{\mathrm{rf}}$ contains an occurrence of an event modality that is not followed by a common knowledge modality or another event modality. The translation function is defined to decompose such formulas, and we see from Proposition 4.6 that such a decomposition either reduces the complexity of the formula unless the formula has only one symbol or acts on subformulas.

Proposition 4.7. *For every formula $\varphi \in \mathcal{L}_{e+Y}$, $c(t(\varphi)) \leq c(\varphi)$. We have $c(t(\varphi)) = c(\varphi)$ if and only if $t(\varphi) = \varphi$.*

Proof. This is a straightforward proof by induction on the complexity of formulas, where each step follows from the definition of the translation or from Proposition 4.6. Every change t makes to a formula reduces its complexity. $\Box$

Proposition 4.8. *The translation t is a computable function.*

Proof. We can prove by induction on the complexity of formulas φ that $t(\varphi)$ is computable, by noticing that every application of t either terminates (in the case of single symbol formulas), acts on subformulas (in the case of formulas whose main connective is not an event modality), or decreases the complexity of the argument (in the case of formulas whose main connective is an event modality). $\Box$

Definition 4.9 (*Syntactic Equivalence*)**.** We now define a relation $\equiv$ on the set Ω^+ of strings of event modalities to be the smallest relation for which the following hold.

- $\lambda \equiv \lambda$, where λ is the empty string.
- $[\vec{\psi}e] \equiv [\vec{\chi}f]$ iff $e = f$ and $\vdash \psi_i \leftrightarrow \chi_i$ for $1 \leq i \leq n$ (where n is the number of event points in the fixed event frame.
- $\alpha\beta \equiv \alpha'\beta'$ iff $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$ for each $\alpha, \alpha', \beta, \beta' \in \Omega$.

Lemma 4.10. *Whenever $\alpha \xrightarrow{\mathbb{B}}{}^{*} \beta$, we have that $c(\alpha\varphi) = c(\beta\varphi)$ for any φ.*

Proof. This is a straightforward induction proof on the length of α. It uses the fact that for every event model is α, the corresponding event model in β has the exact same preconditions. The inductive step uses the following equation $c([\vec{\psi}e]\alpha'\varphi) = (4+|E|+\max\{c(\vec{\psi})\})c(\alpha'\varphi)$, where we can apply the inductive hypothesis to $\alpha'\varphi$. $\Box$

Lemma 4.11. *For any $\alpha \in \Omega$ and $\varphi \in \mathcal{L}_{e+Y}$, $c(\textsc{Pre}(\alpha)) < c(\alpha\varphi)$.*

Proof. We prove by induction on the number of event modalities in α the stronger claim that $c(\textsc{Pre}(\alpha)) + 1 < c(\alpha\varphi)$. For the base case $\alpha = [\vec{\psi}e_i]$, and hence $c(\textsc{Pre}(\alpha)) = c(\psi_i) < (4+|E|+\max(c(\vec{\psi})))c(\varphi)$. For the inductive step, suppose the desired result holds for β. Then

$$
\begin{aligned}
c(\textsc{Pre}([\vec{\psi}e_i]\alpha) &= c(\psi_i \wedge [\vec{\psi}e_i]\textsc{Pre}(\alpha)) \\
&= 1+\max\{c(\psi_i), c([\vec{\psi}e_i]\textsc{Pre}(\alpha))\} \\
&= 1+(4+|E|+\max(c(\vec{\psi})))c(\textsc{Pre}(\alpha)) \\
&< (4+|E|+\max(c(\vec{\psi})))(c(\textsc{Pre}(\alpha))+1) \\
&< (4+|E|+\max(c(\vec{\psi})))c(\alpha\varphi) \\
&= c([\vec{\psi}e_i]\alpha\varphi). \quad \Box
\end{aligned}
$$

We next wish to show that every formula is provably equivalent to its translation.

Theorem 4.12. *For all formulas φ in $\mathcal{L}_{e+Y}$, $\vdash \varphi \leftrightarrow t(\varphi)$.*

Proof. We will prove this by induction on the complexity of φ. Much of the proof is similar to a similar proof in [9], but to help with cases involving common knowledge, we will use a stronger inductive hypothesis that will have two parts.

Inductive hypotheses: Two inductive hypotheses are as follows.

(a) $\vdash \varphi \leftrightarrow t(\varphi)$, whenever $c(\varphi) \leq k$.
(b) $\vdash \alpha\varphi \leftrightarrow \alpha'\varphi$, whenever $\alpha \in \Omega^+$, $\alpha \equiv \alpha'$ and $c(\alpha\varphi), c(\alpha'\varphi) \leq k$.

We list the base case and the inductive step cases below. Some inductive step cases depend on other inductive step cases. The cases are arranged so that no case depends on one that has not yet been proved.

(a) cases γ true ($\gamma \in \Omega^+$), $p, e, \neg\varphi, \varphi \wedge \psi, \bar{Y}\varphi, \Box_A\varphi, \Box^*_{\mathbb{B}}\varphi$**:** These are straightforward.

(a) cases $[\vec{\psi}e]p, [\vec{\psi}e]\neg\varphi, [\vec{\psi}e](\varphi_1 \wedge \varphi_2), [\vec{\psi}e]\Box_A\varphi$**:** These cases are in [9].

(a) case $[\vec{\psi}e_i]e_i$**:** This case is immediate from the fact that $[\vec{\psi}e_i]e_i$ is provably equivalent to true $= t([\vec{\psi}e_i]e_i)$ by the axiom *future event point mix*.

(a) case $[\vec{\psi}e_i]e_j$ $(i \neq j)$**:** This case is immediate from Definition 4.4 and Proposition 3.18.

(a) case $[\vec{\psi}e_i]\bar{Y}\varphi$**:** This case is immediate from Definition 4.4 and axiom *future past mix*.

(a) cases $\gamma[\vec{\psi}e_i]\varphi$, where $\varphi = p, e_i, e_j (j \neq i), \neg\chi, \chi_1 \wedge \chi_2, \Box_A\chi, \bar{Y}\chi$ and $\gamma \in \Omega$**:** For each of these, observe that $t(\gamma[\vec{\psi}e_i]\varphi) = t(\gamma t([\vec{\psi}e_i]\varphi))$. Also, given the cases for what φ can be, $t([\vec{\psi}e_i]\varphi) \neq [\vec{\psi}e_i]\varphi$. Then by Proposition 4.7 that $c(t([\vec{\psi}e_i]\varphi)) < c([\vec{\psi}e_i]\varphi)$, whence $c(\gamma t([\vec{\psi}e_i]\varphi) < c(\gamma[\vec{\psi}e_i]\varphi) = k+1$. Also note that γ is not the empty string λ, and hence $c([\vec{\psi}e_i]\varphi) < c(\gamma[\vec{\psi}e_i]\varphi) = k+1$. We then apply the inductive hypothesis (a) to get both provable equivalences in the following (note that necessitation is also used to establish the first provable equivalence):

$$\gamma[\vec{\psi}e_i]\varphi \equiv \gamma t([\vec{\psi}e_i]\varphi) \equiv t(\gamma t([\vec{\psi}e_i]\varphi)) = t(\gamma[\vec{\psi}e_i]\varphi)$$

(b) case for γ true ($\gamma \in \Omega^+$)**:** This case is immediate.

(b) cases for $\gamma\varphi$, where $\varphi = p, e, \neg\chi, \chi_1 \wedge \chi_2, \Box_A\chi, \bar{Y}\chi$ and $\gamma \in \Omega^+$**:** Assume that $c(\alpha\gamma\varphi), c(\alpha'\gamma\varphi) \leq k+1$. Note that given the cases for what φ can be, $t(\alpha\gamma\varphi) \neq \alpha\gamma\varphi$. Then by Proposition 4.7, $c(t(\alpha\gamma\varphi)) < c(\alpha\gamma\varphi)$ and similarly for $\alpha'\gamma\varphi$. Then

$$\alpha\gamma\varphi \equiv_1 t(\alpha\gamma\varphi) \equiv_2 t(\alpha'\gamma\varphi) \equiv_3 \alpha'\gamma\varphi.$$

The provable equivalences $\equiv_1$ and $\equiv_3$ come from inductive steps (a) already proved above. Note that these (a) cases do not depend on any other cases, only the inductive hypothesis. The provable equivalence $\equiv_2$ is a direct application of the inductive hypothesis (b).

(b) case $\Box^*_{\mathbb{B}}\varphi$**:** Assume $c(\alpha\Box^*_{\mathbb{B}}\varphi), c(\alpha'\Box^*_{\mathbb{B}}\varphi) \leq k+1$. We use the event rule to show that $\vdash \alpha\Box^*_{\mathbb{B}}\varphi \to \alpha'\Box^*_{\mathbb{B}}\varphi$; the other direction uses the same argument. For every β, γ, for which $\alpha \overset{\mathbb{B}}{\to}{}^* \beta$, and $\beta \overset{A}{\to} \gamma$, we have
(1) $\vdash \beta\Box^*_{\mathbb{B}}\varphi \to \beta\varphi$ and
(2) $\vdash \beta\Box^*_{\mathbb{B}}\varphi \wedge \text{PRE}(\beta) \to \Box_A\gamma\Box^*_{\mathbb{B}}\varphi$, for each $A \in \mathbb{B}$.
The justification for these two is from [3] and goes as follows: (1) follows from the axiom *epistemic mix* and modal reasoning. For (2), we start with a consequence of the axiom *epistemic mix*: $\vdash \Box^*_{\mathcal{B}}\varphi \to \Box_A\Box^*_{\mathcal{B}}\varphi$. Then by modal reasoning, $\vdash \beta\Box^*_{\mathcal{B}}\varphi \to \beta\Box_A\Box^*_{\mathcal{B}}\varphi$. Then by the extended event knowledge axiom (Proposition 3.14) and propositional logic, we have $\vdash \beta\Box^*_{\mathcal{B}}\varphi \wedge \text{PRE}(\beta) \to \Box_A\gamma\Box^*_{\mathcal{B}}\varphi$.

A simple induction argument on the length of α gives us the following: for every β' for which $\alpha' \overset{\mathbb{B}}{\to}{}^* \beta'$, there is a unique β for which $\alpha \overset{\mathbb{B}}{\to}{}^* \beta$ and $\beta \equiv \beta'$. Note that $c(\alpha\varphi) < c(\alpha\Box^*_{\mathbb{B}}\varphi) \leq k+1$ and similarly with α'. By Lemma 4.10, $c(\beta\varphi) = c(\alpha\varphi) < k+1$ and similarly with the β'. We can also use Lemma 4.11 to establish that $c(\text{PRE}(\beta)) < c(\beta\varphi) < k+1$, and similarly with β'. Thus we can apply the induction hypothesis (b) to get $\vdash \beta\varphi \leftrightarrow \beta'\varphi$. A straightforward induction on the length of β together using many applications of the inductive hypothesis (b) yields $\vdash \text{PRE}(\beta) \leftrightarrow \text{PRE}(\beta')$. Thus we have
(1′) $\vdash \beta\Box^*_{\mathbb{B}}\varphi \to \beta'\varphi$ and
(2′) $\vdash \beta\Box^*_{\mathbb{B}}\varphi \wedge \text{PRE}(\beta') \to \Box_A\gamma\Box^*_{\mathbb{B}}\varphi$, for each $A \in \mathbb{B}$.
By the event rule, $\vdash \alpha\Box^*_{\mathbb{B}}\varphi \to \alpha'\Box^*_{\mathbb{B}}\varphi$.

(b) case $[\vec{\psi}e]\gamma\Box^*_{\mathbb{B}}\varphi$ $(\gamma \in \Omega^+)$**:** Assume $c(\alpha[\vec{\psi}e]\gamma\Box^*_{\mathbb{B}}\varphi), c(\alpha'[\vec{\psi}e]\gamma\Box^*_{\mathbb{B}}\varphi) \leq k+1$. This is almost identical to the case (b) for $\Box^*_{\mathbb{B}}\varphi$. Replace α in the previous argument with $\alpha[\vec{\psi}e]\gamma$ and α' with $\alpha'[\vec{\psi}e]\gamma$.

(a) case $[\vec{\psi}e]\gamma\Box^*_{\mathbb{B}}\varphi$ $(\gamma \in \Omega^+)$**:** Assume $c([\vec{\psi}e]\gamma\Box^*_{\mathbb{B}}\varphi) = k+1$. Then

$$[\vec{\psi}e]\gamma\Box^*_{\mathbb{B}}\varphi \equiv_1 [t(\vec{\psi})e]t(\gamma\Box^*_{\mathbb{B}}\varphi) \equiv_2 t([\vec{\psi}e]t(\gamma\Box^*_{\mathbb{B}}\varphi)) = t([\vec{\psi}e]\gamma\Box^*_{\mathbb{B}}\varphi).$$

The first provable equivalence $\equiv_1$ comes from the inductive hypothesis (a) and inductive step (b) cases for $\gamma\Box^*_{\mathbb{B}}\varphi$ or $\Box^*_{\mathbb{B}}\varphi$, depending on whether $\gamma = \lambda$ in this step. Note that these two inductive step cases do not make use of any other inductive step case (just the inductive hypotheses). The second provable equivalence $\equiv_2$ comes from Definition 4.4 and the inductive hypothesis (b). $\Box$

4.2. *Epistemic temporal semantics and histories*

Definition 4.13 (*Epistemic Temporal Semantics*)**.** The semantics shall be defined as the smallest relation $\vDash$ between pointed epistemic temporal models and formulas in $\mathcal{L}_{\mathsf{rf}}$ for which the following holds: $\mathcal{M}, x \vDash$ true for all pointed models $(\mathcal{M}, x)$

and

$$
\begin{array}{lll}
\mathcal{M}, x \vDash p & \text{iff} & x \in \|p\| \\
\mathcal{M}, x \vDash e & \text{iff} & e = \varepsilon(x) \\
\mathcal{M}, x \vDash \neg\psi & \text{iff} & \mathcal{M}, x \nvDash \psi \text{ (meaning it is not the case that } \mathcal{M}, x \vDash \psi) \\
\mathcal{M}, x \vDash \psi_1 \wedge \psi_2 & \text{iff} & \mathcal{M}, x \vDash \psi_1 \text{ and } \mathcal{M}, x \vDash \psi_2 \\
\mathcal{M}, x \vDash \bar{Y}\psi & \text{iff} & \mathcal{M}, z \vDash \psi \text{ whenever } xYz \\
\mathcal{M}, x \vDash \Box_A\psi & \text{iff} & \mathcal{M}, z \vDash \psi \text{ whenever } x \xrightarrow{A} z \\
\mathcal{M}, x \vDash \Box^*_{\mathbb{B}}\psi & \text{iff} & \mathcal{M}, z \vDash \psi \text{ whenever } x \xrightarrow{\mathbb{B}}{}^{*} z \\
\mathcal{M}, x \vDash \alpha\Box^*_{\mathbb{B}}\psi & \text{iff} & \begin{cases} \mathcal{M}, x_k \vDash t(\alpha_k\psi) \text{ for all sequences of length } k \geq 0 \\ x = x_0 \xrightarrow{A_1} x_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} x_{k-1} \xrightarrow{A_k} x_k \\ \alpha = \alpha_0 \xrightarrow{A_1}_{\Omega} \alpha_1 \xrightarrow{A_2}_{\Omega} \cdots \xrightarrow{A_{k-1}}_{\Omega} \alpha_{k-1} \xrightarrow{A_k}_{\Omega} \alpha_k \\ \text{such that each } A_i \in \mathbb{B} \text{ and } \mathcal{M}, x_i \vDash t(\textsc{Pre}(\alpha_i)) \text{ for each } i < k. \end{cases}
\end{array}
$$

Note that if $\varepsilon(x) = \emptyset$, there is no event point e which is true at x.

Definition 4.14 (*Characterizable Epistemic Temporal History*). Given an event frame $\mathcal{F}$ and set Φ of atomic propositions, a structure $\mathcal{M} = (S, \overset{\cdot}{\to} \| \cdot \|, Y, \varepsilon)$ is called an *epistemic temporal history* (ETH) if it has the following properties.

1. Event points: $\varepsilon(x) \neq \emptyset$ iff there exists z such that xYz.
2. Partial functionality of Y: If xYz and xYz' then $z = z'$.
3. Bounded age: There exists N such that for all x there is no z for which xY^Nz.
4. Synchronicity: if $x \xrightarrow{A} z$, then for each n, xY^nx' for some x' iff zY^nz' for some z'.
5. States a: If uYz, xYz, and $x \neq u$, then $\varepsilon(u) \neq \varepsilon(x)$. We may view this property using the following diagram, where solid arrows are assumed in the premise and the dotted line expresses the conclusion. If $u \neq x$ then

$$
\begin{tikzcd}
u \arrow[rr, dash, dashed, "\varepsilon(u)\neq\varepsilon(x)"] \arrow[dr, "Y"'] & & x \arrow[dl, "Y"] \\
& z &
\end{tikzcd}
$$

6. States b: For each event point $e \in E$ and each $k \geq 0$, there exists a formula $\varphi \in \mathcal{L}_{\mathrm{rf}}$ such that for every x for which xY^kz for some z and $xY^{k+1}z$ for no z,

   $\mathcal{M}, x \vDash \varphi$ iff there is a u such that $\varepsilon(u) = e$ and uYx.

7. Relation a: If $x \xrightarrow{A} zYz'$, there exists an x' such that $xYx' \xrightarrow{A} z'$. We may view this using the following diagram, where the solid arrows are assumed in the premise, and the dotted arrow is the conclusion.

$$
\begin{tikzcd}
x \arrow[r, "A"] \arrow[d, "Y"'] & z \arrow[d, "Y"] \\
x' \arrow[r, dashed, "A"] & z'
\end{tikzcd}
$$

   (Considering the contrapositive, we see that certainty in the past translates to certainty in the present, which is the perfect recall condition.)
8. Relation b: If $x \xrightarrow{A} z$, and $\varepsilon(x) \neq \emptyset$, then $\varepsilon(x) \xrightarrow{A}_{\mathcal{F}} \varepsilon(z)$. (This is related to uniform no-miracles.)
9. Relation c: If xYx', zYz', $x' \xrightarrow{A} z'$, and $\varepsilon(x) \xrightarrow{A} \varepsilon(z)$, then $x \xrightarrow{A} z$. (This is related to uniform no-miracles.) We may view this property using the following diagram, where solid arrows are part of the premise and the dotted arrow is part of the conclusion. If $\varepsilon(x) \xrightarrow{A} \varepsilon(z)$

$$
\begin{tikzcd}
x \arrow[r, dashed, "A"] \arrow[d, "Y"'] & z \arrow[d, "Y"] \\
x' \arrow[r, "A"] & z'
\end{tikzcd}
$$

10. Valuation: If xYz, then $\mathcal{M}, x \vDash p$ iff $\mathcal{M}, z \vDash p$.

An epistemic temporal model satisfying all conditions except *states b* of a characterizable epistemic temporal history is called an *epistemic temporal history* that is not necessarily characterizable. This notion will be helpful in establishing the proof of Lemma 4.16, which will show that a ETH is isomorphic to a sequential history, using the following notion of isomorphism.

Definition 4.15 (*Isomorphism*)**.** A ETM $\mathcal{M} = (X, \to, Y, g, \|\cdot\|)$ is *isomorphic* to a state model sequence $\mathcal{H} = (\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n)$ with $\mathcal{S}_k = (S_k, \to_k, \|\cdot\|_k)$ if there exists a bijective function f from X to $\bigcup_{k=0}^{n} S_k$, such that

1. $x \in \|p\|$ iff $f(x) \in \|p\|_k$ if $f(k) \in S_k$.
2. $\varepsilon(x) = e$ iff $f(x) = (y, e)$ for some $y \in S_k$ with $0 \le k < n$.
3. $x \xrightarrow{A} z$ iff $f(x) \xrightarrow{A}_k f(z)$ if $f(x) \in S_k$.
4. xYz iff $f(x) = (f(z), g(x))$.

The function f is called an *isomorphism*.

Lemma 4.16. *Every ETH is isomorphic to some sequential history.*

Proof. Suppose $\mathcal{M} = (S_H, \xrightarrow{\mathbb{A}}_H, Y_H, g_H, \|\cdot\|_H)$ is an ETH for an event frame $\mathcal{F}$ and set Φ of atomic propositions. Let

$$S_0 = \{x \in S_H : \varepsilon(x) = \emptyset\}.$$

By *event points* (condition 1) and *partial functionality of* Y (condition 2), we can uniquely map every state $x \notin S_0$ to (z, e), where xYz and $e = \varepsilon(x)$. Using this, let

$$f(x) = \begin{cases} x & x \in S_0 \\ (f(z), e) & xYz,\, e = \varepsilon(x). \end{cases}$$

Condition 5, *states a*, ensures that f is injective. We will partition the image of f into the carrier sets of state models that form a sequential history. For each $k > 0$, let

$$S_k = \{f(x) : xY^k z \text{ for some } z \in S, \text{ and } xY^{k+1}z \text{ for no } z\}.$$

By *bounded age* (condition 3), there is an N such that for every $x \in \mathcal{M}$, $f(x)$ will be in some S_k for $k \le N$. By *synchronicity* (condition 4), we know that if $f(x) \in S_k$ and $x \xrightarrow{A} z$ then $f(z) \in S_k$. Then for each set S_k, we define $\mathcal{S}_k = (S_k, \to_k, \|\cdot\|_k)$, where

1. $f(x) \xrightarrow{A}_k f(z)$ if both $x \xrightarrow{A}_H z$ and $f(x) \in S_k$ (and hence $f(z) \in S_k$).
2. $\|p\|_k = S_k \cap \{f(x) : x \in \|p\|_H\}$.

We now have a sequence of state models.

Let us construct a list of alleged sequential histories as follows: $\mathcal{H}_0 = (\mathcal{S}_0)$ and for each k, $\mathcal{H}_{k+1} = (\mathcal{S}_0, \ldots, \mathcal{S}_{k+1})$. Finally, we let $\mathcal{H} = \mathcal{H}_N$. To see that each $\mathcal{H}_k$ is a history, we must show that each follows from the previous from some update product. We do not claim in this lemma that we are proving that this sequential history is characterizable. But we will later use *states b* (condition 6) and Lemma 4.17 to show that if $\mathcal{M}$ were characterizable, then this sequential history would be too. The relation condition of an update product requires that if (s, e) and (t, g) both exist, then $(s, e) \xrightarrow{A} (t, g)$ iff both $s \xrightarrow{A} t$ and $e \xrightarrow{A}_{\mathcal{F}} g$. Observe that *relation a* and *relation b* (conditions 7 and 8) give us the *only if*, and *relation c* (condition 9) gives us the *if*. Finally, *valuation* (condition 10) guarantees us the valuation condition of the update product.

The isomorphism properties for f are immediate from the construction of the sequential history $\mathcal{H}$. □

We finally see that this isomorphism preserves truth of normal form formulas.

Lemma 4.17. *Given an epistemic temporal history $\mathcal{M}$, a sequential history $\mathcal{H}$, an isomorphism f from $\mathcal{M}$ to H, any $x \in \mathcal{M}$, and any formula $\varphi \in \mathcal{L}_{rf}$,*

$$\mathcal{M}, x \vDash \varphi \quad \textit{iff} \quad f(x) \in [\![\varphi]\!](prvh^k(\mathcal{H})),$$

where k is such that $f(x) \in prvh^k(\mathcal{H})$.

Proof. Fix an epistemic temporal history $\mathcal{M} = (S_M, \to_M, Y_M, \varepsilon_M, \|\cdot\|_M)$ and a sequential history $\mathcal{H}$. Let $\mathcal{H} = (\mathcal{S}_0, \ldots, \mathcal{S}_N)$, where for $0 \le k \le N$, $\mathcal{H}_k = (\mathcal{S}_0, \ldots, \mathcal{S}_k)$ and $\mathcal{S}_k = (S_k, \to_k, \|\cdot\|_k)$.

We prove the desired result by induction on the complexity of the formula φ.

Inductive hypothesis: $\mathcal{M}, x \vDash \varphi$ iff $f(x) \in [\![\varphi]\!](\mathcal{H}_k)$ for each x for which $f(x) \in S_k$, and each $\varphi \in \mathcal{L}_{\mathrm{rf}}$ for which $c(\varphi) \le j$.

base cases true, p, e**:** These are straightforward from definitions. Note that $\varepsilon(x) = e$ iff $f(x) = (y, e)$ for some $y \in S_k$ with $0 \le k < n$.

cases $\neg\psi$ and $\psi_1 \wedge \psi_2$**:** These are also straightforward.

case $\bar{Y}\psi$**:** We first consider x such that $f(x) \in S_0$. By definition of f, there is no z such that xYz, and hence $\mathcal{M}, x \vDash \bar{Y}\psi$. As $\mathcal{H}_0$ has no past, $f(x) \in [\![\bar{Y}\psi]\!](\mathcal{H}_0)$.

Next suppose $f(x) \in S_k$ for $k > 0$. By the property xYz iff $f(x) = (f(z), g(x))$, there is exactly one z for which xYz. Hence the following are equivalent:

1. $\mathcal{M}, x \vDash \bar{Y}\psi$

2. $\mathcal{M}, z \vDash \psi$
3. $f(z) \in [\![\psi]\!](\mathcal{H}_{k-1})$
4. $f(x) \in [\![\bar{Y}\psi]\!](\mathcal{H}_k)$

cases $\Box_A\psi$ and $\Box^*_{\mathbb{B}}\psi$**:** These come from the tight connection between $\xrightarrow{A}$ in $\mathcal{M}$ and $\xrightarrow{A}_k$ in each $\mathbf{S}_k$. Note that if $x \xrightarrow{A}_M z$, then $f(x) \xrightarrow{A}_k f(z)$ for some k, and hence $f(x)$ and $f(z)$ are in the same state model.

case $\alpha\Box^*_{\mathbb{B}}\psi$ for $\alpha \in \Omega$**:** First suppose that $\mathcal{M}, x \nvDash \alpha\Box^*_{\mathbb{B}}\psi$. Then there is a sequence

$$x = x_0 \xrightarrow{A_1} x_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} x_{k-1} \xrightarrow{A_k} x_k$$

in $\mathcal{M}$ such that $k \geq 0$, each $A_i \in \mathbb{B}$, and a sequence

$$\alpha = \alpha_0 \xrightarrow{A_1}_{\Omega} \alpha_1 \xrightarrow{A_2}_{\Omega} \cdots \xrightarrow{A_{k-1}}_{\Omega} \alpha_{k-1} \xrightarrow{A_k}_{\Omega} \alpha_k$$

such that $\mathcal{M}, x_i \vDash t(\textsc{Pre}(\alpha_i))$ for each $0 \leq i \leq k$, and $\mathcal{M}, x_k \vDash t(\neg\alpha_k\psi)$. To apply the inductive hypothesis, we observe that $c(t(\textsc{Pre}(\alpha_i))) \leq c(\textsc{Pre}(\alpha_i)) < c(\alpha_i\psi) = c(\alpha\psi) < c(\alpha\Box^*_{\mathbb{B}}\psi)$ and $c(t(\neg\alpha_k\psi)) \leq c(\neg\alpha_k\psi) = c(\neg\alpha\psi) < c(\alpha\Box^*_{\mathbb{B}}\psi)$ (note that $\alpha \neq \lambda$), by Proposition 4.7 and Lemmas 4.10 and 4.11. After applying the inductive hypotheses, we use Lemma 3.10. The other direction is parallel to this one. $\Box$

Lemma 4.18. *Every characterizable ETH is isomorphic to some characterizable sequential history.*

Proof. In the proof of Lemma 4.16, we constructed a sequence of state models $\mathcal{S}_k$ and histories $\mathcal{H}_k$. We wish to show that $\mathcal{H}_{k+1} = \mathcal{H}_k \otimes [\![\vec{\varphi}]\!]$. For each $k \geq 0$, we have by *states b* (condition 6) a formula φ_i for each e_i, such that for every x where xY^kz for some z and $xY^{k+1}z$ for no z,

$\mathcal{M}, x \vDash \varphi_i$ iff there is a w such that $\varepsilon(w) = e_i$ and wYx.

We let $\vec{\varphi}$ be the list of each of these formulas. Our desired result follows from Lemma 4.17. $\Box$

4.3. Bisimulations

Definition 4.19 (*Bisimulation between ETMs*)**.** A bisimulation between ETMs $\mathcal{M}_1 = (S_1, \rightarrow_1, Y_1, \varepsilon_1, \|\cdot\|_1)$ and $\mathcal{M}_2 = (S_2, \rightarrow_2, Y_2, \varepsilon_2, \|\cdot\|_2)$ is a relation $R \subseteq S_1 \times S_2$ such that the following hold.

1. if xRz then for all $p \in \Phi$, $x \in \|p\|_1$ iff $z \in \|p\|_2$.
2. if xRz then $\varepsilon_1(x) = \varepsilon_2(z)$.
3. if xRz and $x \xrightarrow{A}_1 x'$, then there exists z' such that $x'Rz'$ and $z \xrightarrow{A}_2 z'$.
4. if xRz and $z \xrightarrow{A}_2 z'$, then there exists x' such that $x'Rz'$ and $z \xrightarrow{A}_1 z'$.
5. if xRz and xY_1x', then there exists z' such that $x'Rz'$ and zY_2z'.
6. if xRz and zY_2z', then there exists x' such that $x'Rz'$ and zY_1z'.

Two states x and z are said to be *bisimilar* if xRz for some bisimulation R. If there is a bisimulation between two ETMs, they are said to be *bisimilar*.

Lemma 4.20. *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two ETMs, $x \in S_1$, $z \in S_2$, and x and z are bisimilar, then for every $\psi \in \mathcal{L}_{rf}$,*

$\mathcal{M}_1, x \vDash \psi$ *iff* $\mathcal{M}_2, z \vDash \psi$

Proof. Let R be a bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ for which xRz. We prove this by induction on the complexity of the formula.

Inductive hypothesis: $\mathcal{M}_1, a \vDash \varphi$ iff $\mathcal{M}_2, b \vDash \varphi$, whenever $c(\varphi) \leq k$ and aRb.

Base cases true, p, e**:** These come directly from the definition of bisimulation.

Cases $\neg\psi$, $\psi_1 \wedge \psi_2$, $\bar{Y}\psi$, $\Box_A\psi$**:** These use standard modal arguments.

Case $\Box^*_{\mathbb{B}}\psi$**:** Suppose that $\mathcal{M}_1, x \nvDash \Box^*_{\mathbb{B}}\psi$. Then $\mathcal{M}_1, x \vDash \neg\Diamond^*_{\mathbb{B}}\psi$. Then there exists a path of states connected by epistemic relations for agents in $\mathbb{B}$ from x to another state x' for which $\neg\psi$ is true. Repeating property (3) of the bisimulation gives us a path using the same relation from z to z' for which $x'Rz'$. We apply the inductive hypothesis to determine that $\mathcal{M}_2, z' \vDash \neg\psi$, and hence $\mathcal{M}_2, z \nvDash \Box^*_{\mathbb{B}}\psi$. The converse is similar, except we use property (4) rather than (3).

Case $\alpha\Box^*_{\mathbb{B}}\psi$ ($\alpha \in \Omega$): Suppose that $\mathcal{M}_1, x \nvDash \alpha\Box^*_{\mathbb{B}}\psi$. Then there exists a sequence

$$x = x_0 \xrightarrow{A_1} x_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} x_{k-1} \xrightarrow{A_k} x_k$$

in $\mathcal{M}_1$ such that $k \geq 0$, each $A_i \in \mathbb{B}$ and a sequence

$$\alpha = \alpha_0 \xrightarrow{A_1}_{\Omega} \alpha_1 \xrightarrow{A_2}_{\Omega} \cdots \xrightarrow{A_{k-1}}_{\Omega} \alpha_{k-1} \xrightarrow{A_k}_{\Omega} \alpha_k$$

such that $\mathcal{M}_1, x_i \vDash t(\textsc{Pre}(\alpha_i))$ for each $0 \leq i \leq k$, and $\mathcal{M}_1, x_k \vDash t(\neg\alpha_k\psi)$. Similar to the $\Box^*_{\mathbb{B}}$ case, we construct a path in $\mathcal{M}_2$ from z to a state z_k. To apply the inductive hypothesis, we observe that $c(t(\textsc{Pre}(\alpha_i))) \leq c(\textsc{Pre}(\alpha_i)) < c(\alpha_i\psi) < c(\alpha\Box^*_{\mathbb{B}}\psi)$ and $c(t(\neg\alpha_k\psi)) \leq c(\neg\alpha_k\psi) < c(\alpha\Box^*_{\mathbb{B}}\psi)$, by Proposition 4.7 and Lemmas 4.10 and 4.11. The converse uses a parallel argument. $\Box$

5. Filtration and model shaping

5.1. Generating (Closure) function

In the completeness proof, we will fix a consistent formula in $\mathcal{L}_{e+Y}$ that is provably equivalent to formula $\varphi \in \mathcal{L}_{\mathsf{rf}}$. We will construct a characterizable Epistemic Temporal History (ETH) that satisfies φ, by first constructing a filtration that is an Epistemic Temporal Model (ETM), and then modifying the model to one that is a characterizable ETH. To create a filtration, we first define a closure function $cl : \mathcal{L}_{\mathsf{rf}} \to \mathcal{P}(\mathcal{L}_{\mathsf{rf}})$. We will then perform four transformations. First we will pick one state $[U]$ in the filtration which satisfies φ, and unravel the filtration with $[U]$ as the root. A bisimulation will be established, and hence the truth lemma for the filtration can be transferred to the unraveled model. Another transformation will be made to trim the relation Y so that it is both a partial function bounded by the yesterday depth of the formula φ, which we define as follows:

Definition 5.1 (*Yesterday Depth Function*). Define $\mathsf{dep} : \mathcal{L}_{\mathsf{rf}} \to \mathbb{Z}$ to be a function mapping $\mathcal{L}_{\mathsf{rf}}$ to non-negative integers as follows:

1. $\mathsf{dep}(\mathsf{true}) = \mathsf{dep}(p) = 0$
2. $\mathsf{dep}(e) = 1$
3. $\mathsf{dep}(\neg\varphi) = \mathsf{dep}(\Box_A\varphi) = \mathsf{dep}(\Box^*_{\mathbb{B}}\varphi) = \mathsf{dep}(\varphi)$
4. $\mathsf{dep}(\varphi \wedge \psi) = \max\{\mathsf{dep}(\varphi), \mathsf{dep}(\psi)\}$
5. $\mathsf{dep}(\bar{Y}\varphi) = \mathsf{dep}(\varphi) + 1$
6. $\mathsf{dep}([\psi_1, \ldots, \psi_n, e]\varphi) = \max\{\mathsf{dep}(\psi_1), \ldots, \mathsf{dep}(\psi_n), \mathsf{dep}(\varphi)\}$.

The next transformation will add epistemic relational connections to ensure the relation property of the update product holds. The trimming transformation and the transformation of adding epistemic operators will not establish a bisimulation between the old and new models, but it will ensure that formulas in $cl(\varphi)$ with yesterday depth k will be preserved at states within k Y-relational steps from a Y-terminal state.

Definition 5.2 (*Effective Negation*). For any formula φ that is not a negation, let $\sim\varphi$ represent $\neg\varphi$ and let $\sim\neg\varphi$ represent φ.

Definition 5.3 (*Closure Function*). Let $cl : \mathcal{L}_{\mathsf{rf}} \to \mathcal{P}(\mathcal{L}_{\mathsf{rf}})$ be defined where $cl(\varphi)$ is the smallest set for which the following hold.

1. $\varphi \in cl(\varphi)$.
2. $\mathsf{true} \in cl(\varphi)$.
3. $\{\Box_A\bar{Y}^k\,\mathsf{false}, \Box_A\neg\bar{Y}^k\,\mathsf{false} : A \in \mathbb{A}, 1 \leq k \leq \mathsf{dep}(\varphi)\} \subset cl(\varphi)$.
4. If $p \in cl(\varphi)$ and $\mathsf{dep}(\varphi) > 0$, then $\bar{Y}p \in cl(\varphi)$.
5. If $\mathsf{dep}(\varphi) > 0$, then $\{e, \Box_A\neg e : e \in E, A \in \mathbb{A}\} \subset cl(\varphi)$.
6. If $\psi \in cl(\varphi)$, then $\mathsf{sub}(\psi) \subseteq cl(\varphi)$.
7. If $\psi \in cl(\varphi)$, and ψ is not a negation, then $\neg\psi \in cl(\varphi)$.
8. If $\Box_A\psi \in cl(\varphi)$ and $\mathsf{dep}(\varphi) > \mathsf{dep}(\Box_A\psi)$, then $\{\bar{Y}\Box_A\psi, \Box_A\bar{Y}\psi\} \subset cl(\varphi)$.
9. If $\Box^*_{\mathbb{B}}\psi \in cl(\varphi)$, then $\{\Box_A\Box^*_{\mathbb{B}}\psi : A \in \mathbb{B}\} \subset cl(\varphi)$.
10. If $\alpha\Box^*_{\mathbb{B}}\psi \in cl(\varphi)$ ($\alpha \in \Omega$), then $\{t(\textsc{Pre}(\beta)), t(\beta\psi) : \alpha \xrightarrow{\mathbb{B}}{}^* \beta\} \subset cl(\varphi)$ and $\{\Box_A\beta\Box^*_{\mathbb{B}}\psi : \alpha \xrightarrow{\mathbb{B}}{}^* \beta, A \in \mathbb{B}\} \subset cl(\varphi)$.
11. If $\bar{Y}\varphi \in cl(\varphi)$, then $\bar{Y} \sim\varphi \in cl(\varphi)$.

Lemma 5.4. *Given $\varphi \in \mathcal{L}_{rf}$, $cl(\varphi)$ is a finite set of formulas in $\mathcal{L}_{rf}$ that is computable from φ.*

Proof. We show this by induction on the complexity of the formula φ.

Inductive hypothesis: Suppose $cl(\varphi)$ is finite whenever $\varphi \in \mathcal{L}_{\mathrm{rf}}$ is such that $c(\varphi) \leq k$.
Base case true: $cl(\text{true}) = \{\text{true}, \neg\,\text{true}\}$.
Base case p: $cl(p) = \{p, \neg p\}$.
Base case e:

$$\begin{aligned} cl(e) &= \{e, \neg e\} \cup \{\bar{Y}\,\text{true}, \neg\bar{Y}\,\text{true}, \bar{Y}\,\text{false}, \neg\bar{Y}\,\text{false}\} \\ &\quad \cup \{\Box_A\neg e, \neg\Box_A\neg e, \Box_A\bar{Y}\,\text{false}, \neg\Box_A\neg\bar{Y}\,\text{false} : A \in \mathbb{A}\}, \end{aligned}$$

which is finite as $\mathbb{A}$ is finite.
Case $\neg\varphi$: $cl(\neg\varphi) = \{\neg\varphi\} \cup cl(\varphi)$.
Case $\varphi_1 \wedge \varphi_2$: $cl(\varphi_1 \wedge \varphi_2) = \{\varphi_1 \wedge \varphi_2, \neg(\varphi_1 \wedge \varphi_2)\} \cup cl(\varphi_1) \cup cl(\varphi_2)$.
Case $\bar{Y}\varphi$:

$$\begin{aligned} & cl(\bar{Y}\varphi) = cl(\varphi) \\ & \cup\{\neg\bar{Y}\varphi, \neg\bar{Y}\sim\varphi, \bar{Y}\sim\varphi, \sim\varphi\} \\ & \cup\{\Box_A\bar{Y}^{k+1}\,\text{false}, \neg\Box_A\bar{Y}^{k+1}\,\text{false}, \Box_A\neg\bar{Y}^{k+1}\,\text{false}, \neg\Box_A\neg\bar{Y}^{k+1}\,\text{false} : k = \mathsf{dep}(\varphi)\} \\ & \cup\{\bar{Y}^{k+1}\,\text{false}, \neg\bar{Y}^{k+1}\,\text{false}, \bar{Y}\neg\bar{Y}^{k}\,\text{false}, \neg\bar{Y}\neg\bar{Y}^{k}\,\text{false} : k = \mathsf{dep}(\varphi)\} \\ & \cup\{\bar{Y}p, \neg\bar{Y}p, \bar{Y}\neg p, \neg\bar{Y}\neg p : p \in cl(\varphi)\} \\ & \cup\{\bar{Y}\Box_A\psi, \bar{Y}\neg\Box_A\psi, \Box_A\bar{Y}\psi : \Box_A\psi \in cl(\varphi), A \in \mathbb{A}\} \\ & \cup\{\neg\bar{Y}\Box_A\psi, \neg\bar{Y}\neg\Box_A\psi, \neg\Box_A\bar{Y}\psi : \Box_A\psi \in cl(\varphi), A \in \mathbb{A}\}. \end{aligned}$$

Case $\Box_A\varphi$: $cl(\Box_A\varphi) = \{\Box_A\varphi, \neg\Box_A\varphi\} \cup cl(\varphi)$
Case $\Box^*_{\mathbb{B}}\varphi$: $cl(\Box^*_{\mathbb{B}}\varphi) = \{\Box_A\Box^*_{\mathbb{B}}\varphi, \neg\Box_A\Box^*_{\mathbb{B}}\varphi : A \in \mathbb{A}\} \cup \{\Box^*_{\mathbb{B}}\varphi, \neg\Box^*_{\mathbb{B}}\varphi\} \cup cl(\varphi)$.
Case $\alpha\Box^*_{\mathbb{B}}\varphi$ ($\alpha \in \Omega$): Suppose $c(\alpha\Box^*_{\mathbb{B}}\varphi) = k + 1$.

$$\begin{aligned} & cl(\alpha\Box^*_{\mathbb{B}}\varphi) \\ & = \{\Box_A\beta\Box^*_{\mathbb{B}}\varphi, \neg\Box_A\beta\Box^*_{\mathbb{B}}\varphi, \beta\Box^*_{\mathbb{B}}\varphi, \neg\beta\Box^*_{\mathbb{B}}\varphi : A \in \mathbb{B}, \alpha \overset{\mathbb{B}}{\rightarrow}{}^{*}\ \beta\} \\ & \cup \{\Box_A\Box^*_{\mathbb{B}}\varphi, \neg\Box_A\Box^*_{\mathbb{B}}\varphi : A \in \mathbb{A}\} \cup \{\Box^*_{\mathbb{B}}\varphi, \neg\Box^*_{\mathbb{B}}\varphi\} \cup cl(\varphi) \\ & \cup \bigcup\{cl(t(\textsc{Pre}(\beta))) : \alpha \overset{\mathbb{B}}{\rightarrow}{}^{*}\ \beta\}. \end{aligned}$$

Note that there are finitely many β for which $\alpha \overset{\mathbb{B}}{\rightarrow}{}^{*}\ \beta$, the number of such β being bounded by the product of the length of α and the size of the set E of event points in the fixed event frame. It is immediate from the definition of t, that $c(\alpha\Box^*_{\mathbb{B}}\varphi) > c(\alpha\varphi)$. By Lemma 4.10, for each φ, $c(\beta\varphi) = c(\alpha\varphi)$, by Lemma 4.11 $c(\textsc{Pre}(\beta)) < c(\beta\varphi)$, and by Proposition 4.7, $c(t(\textsc{Pre}(\beta))) < c(\textsc{Pre}(\beta))$. Thus we are able to apply the inductive hypothesis.

To see that cl is a computable function, note that the cases in the inductive proof above provide a clear algorithm for computation. To see that $cl(\varphi) \subseteq \mathcal{L}_{\mathrm{rf}}$, one can check that strings of event modalities are only introduced after operators of the form $\Box^*_{\mathbb{B}}$ and that $\mathcal{L}_{\mathrm{rf}}$ is closed under subformulas and under the attachment of non-event modality operators. $\Box$

In completeness proof, we will need a truth lemma that asserts that states a certain yesterday distance from one particular state must satisfy the formulas in its equivalence class that are also in a set of formulas that depends on this distance. The sets $cl_k(\varphi)$'s in the next definition will be those sets of formulas, and $\mathsf{dep}(\varphi) - k$ will be the distance.

Definition 5.5 (*Functions for Layersets*). For each $k \leq \mathsf{dep}(\varphi)$, define $cl_k(\varphi) = \{\psi \in cl(\varphi) : \mathsf{dep}(\psi) \leq k\}$.

Proposition 5.6. $cl_{dep(\varphi)}(\varphi) = cl(\varphi)$

Proof. The rules that explicitly involve event points e or operators $\bar{Y}$ in Definition 5.3 are 3, 4, 5, 8, and 11, and each of these rules either adds formulas with yesterday depth no greater than one already in $cl(\varphi)$ or has conditions ensuring that any formula added to $cl(\varphi)$ will have a yesterday depth that does not exceed that of φ. As for rule 10, which involves strings of event modalities, note that whenever $\alpha \overset{\mathbb{B}}{\rightarrow}{}^{*}\ \beta$, α and β only differ in the event points in each event modality, and hence α and β contribute the same to the yesterday depth. Finally note that the translation t does not affect the yesterday depth of a formula. $\Box$

5.2. Filtration

Definition 5.7 (*Equivalence of Maximal Consistent Sets*). For a formula $\varphi \in \mathcal{L}_{\mathrm{rf}}$, suppose $cl(\varphi) = \{\psi_1, \ldots, \psi_n\}$. We define, for each maximal consistent set U of formulas in $\mathcal{L}_{\mathrm{rf}}$, the formula U^* to be the conjunction $\pm\psi_i \wedge \cdots \wedge \pm\psi_n$, where the sign is determined by membership in U. We define an equivalence relation $\equiv$ over maximal consistent sets by $U \equiv V$ iff $U^* = V^*$. For each maximal consistent set U, we shall denote its equivalence class by $[U]$.

The formula U^* and relation $\equiv$ both depend on φ, but we do not reflect this in the notation, as we will assume U^* and $\equiv$ to be fixed as φ is fixed. We define our filtration assuming that $\mathtt{dep}(\varphi) > 0$. If $\mathtt{dep}(\varphi) = 0$, we could use the exact same argument as given for the proof of the completeness of DEL in [3].

Definition 5.8 (*Structure $\mathcal{M}_F$ (filtration)*)**.** We define the filtration $\mathcal{M}_F$ to be the tuple $(S_F, \rightarrow_F, Y_F, g_F, \|\cdot\|_F)$, defined such that

1. $S_F = \{[U] : U$ is a maximally consistent set$\}$,
2. $[U] \xrightarrow{A}_F [V]$ iff whenever $\Box_A\psi \in U \cap cl(\varphi)$, then also $\psi \in V$,
3. $[U]Y_F[V]$ iff whenever $\bar{Y}\psi \in U \cap cl(\varphi)$, then also $\psi \in V$,
4. $\varepsilon_F : S \to E \cup \{\emptyset\}$ is defined by $\varepsilon_F([U]) = e$ iff $e \in U \cap cl(\varphi)$,
5. $\|p\|_F = \{[U] : p \in U \cap cl(\varphi)\}$.

It is important to note that ε_F is a well-defined function. This fact comes from the definition of *cl* and the axiom *uniqueness of event points*.

Proposition 5.9 (*Existence Lemma for $\xrightarrow{A}_F$*)**.** *If $U^* \wedge \Diamond_A V^*$ is consistent, then $[U] \xrightarrow{A} [V]$.*

Proof. The proof depends on the definition of the relation in S_F, rather than the exact nature of the equivalence classes. Most of this proof is from [3]. Assume $\Box_A\psi \in U \cap cl(\varphi)$, and toward a contradiction that $\psi \notin V$. Since $\psi \in cl(\varphi)$ and $\neg\psi \in V$, we have $\vdash V^* \to \neg\psi$. Thus, $\vdash \Diamond_A V^* \to \Diamond_A\neg\psi$, and so $\vdash U^* \wedge \Diamond_A V^* \to \Box_A\psi \wedge \Diamond_A\neg\psi$. Hence $U^* \wedge \Diamond_A V^*$ is inconsistent. $\Box$

We use the following definition from [3]:

Definition 5.10 (*Good Path*)**.** Suppose $\neg\alpha\Box^*_{\mathbb{B}}\psi \in \mathcal{L}_{\mathrm{rf}}$. A *good path* in $\mathcal{M}_F$ from $[V_0]$ for $\neg\alpha\Box^*_{\mathbb{B}}\psi$ is a path

$$[V_0] \xrightarrow{A_1} [V_1] \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} [V_{k-1}] \xrightarrow{A_k} [V_k]$$

in $\mathcal{M}_F$, such that $k \geq 0$, each $A_i \in \mathbb{B}$, and a sequence

$$\alpha = \alpha_0 \xrightarrow{A_1} \alpha_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} \alpha_{k-1} \xrightarrow{A_k} \alpha_k$$

such that $t(\mathrm{P{\small RE}}(\alpha_i)) \in V_i$, for all $0 \leq i \leq k$, and $t(\neg\alpha_k\psi) \in V_k$.

Lemma 5.11. *Let $[\alpha]\Box^*_{\mathbb{B}}\psi \in cl(\varphi)$. If there is a good path from $[V_0]$ for $\neg\alpha\Box^*_{\mathbb{B}}\psi$, then $\neg\alpha\Box^*_{\mathbb{B}}\psi \in V_0$.*

Proof. This proof is almost the same as the one in [3]. We prove this by induction on the length k of the path. If $k = 0$, then $t(\neg\alpha\psi) \in V_0$. If $\neg\alpha\Box^*_{\mathbb{B}}\psi \notin V_0$, then $\alpha\Box^*_{\mathbb{B}}\psi \in V_0$. Because $\alpha\Box^*_{\mathbb{B}}\psi \in cl(\varphi)$ and $\alpha\Box^*_{\mathbb{B}}\psi \to \alpha\psi$ is provable, we have $t(\alpha\psi) \in V_0$, a contradiction.

Assume the result for k, and suppose that there is a good path from $[V_0]$ for $\neg\alpha\Box^*_{\mathbb{B}}\psi$ of length $k+1$. Using the notation in Definition 5.10, there is a good path of length k from $[V_1]$ for $\neg\alpha_1\Box^*_{\mathbb{B}}\psi$. By the definition of *cl*, we have that $\alpha_1\Box^*_{\mathbb{B}}\psi \in cl(\varphi)$. By the inductive hypothesis, $\neg\alpha_1\Box^*_{\mathbb{B}}\psi \in V_1$.

If $\neg\alpha\Box^*_{\mathbb{B}}\psi \notin V_0$, then $\alpha\Box^*_{\mathbb{B}}\psi \in V_0$. From the axiom *epistemic mix*, we have $\vdash \Box^*_{\mathbb{B}}\psi \to \Box_A\Box^*_{\mathbb{B}}\psi$. By modal reasoning, we obtain $\vdash \alpha\Box^*_{\mathbb{B}}\psi \to \alpha\Box_A\Box^*_{\mathbb{B}}\psi$. By Lemma 3.14, $\vdash \alpha\Box_A\varphi \leftrightarrow (\mathrm{P{\small RE}}(\alpha) \to \bigwedge\{\Box_A\beta\varphi : \alpha \xrightarrow{A}_{\Omega} \beta\})$. Hence we have $\alpha\Box^*_{\mathbb{B}}\psi \wedge \mathrm{P{\small RE}}(\alpha) \to \Box_A\beta\Box^*_{\mathbb{B}}\psi$. As V_0 is a maximal consistent set, V_0 contains $\alpha\Box^*_{\mathbb{B}}\psi \wedge t(\mathrm{P{\small RE}}(\alpha)) \to \Box_A\alpha\Box^*_{\mathbb{B}}\psi$. Thus V_0 contains $\Box_A\alpha\Box^*_{\mathbb{B}}\psi$. By the definition of *cl*, this formula also belongs to $cl(\varphi)$. By the definition of $\xrightarrow{A}_F$, we see that $\alpha_1\Box^*_{\mathbb{B}}\psi \in V_1$. This contradicts our observation at the end of our last paragraph. $\Box$

Lemma 5.12 (*Existence Lemma for Good Path*)**.** *If $V_0^* \wedge \neg\alpha\Box^*_{\mathbb{B}}\psi$ is consistent, then there is a good path from $[V_0]$ for $\neg\alpha\Box^*_{\mathbb{B}}\psi$.*

Proof. This proof makes use of the fact that the filtration is finite. This proof is almost the same as the one in [3]. For each β such that $\alpha \xrightarrow{\mathbb{B}}{}^* \beta$, let S_β be the (finite) set of all $[W] \in \mathcal{M}_F$ such that there is no good path from $[W]$ for $\neg\beta\Box^*_{\mathbb{B}}\psi$. We need to see that $[V_0] \notin S_\alpha$. Suppose toward a contradiction that $[V_0] \in S_\alpha$. Let

$$\chi_\beta = \bigvee\{W^* : [W] \in S_\beta\}.$$

Note that $\neg\chi_\beta$ is provably equivalent to $\bigvee\{X^* : [X] \in \mathcal{M}_F$ and $[X] \notin S_\beta\}$. Since we assumed $[V_0] \in S_\alpha$, we have $\vdash V_0^* \to \chi_\alpha$.

We first claim that for β such that $\alpha \xrightarrow{\mathbb{B}}{}^* \beta$, $\chi_\beta \wedge \neg\beta\psi$ is inconsistent. Otherwise, there would be $[W] \in S_\beta$ such that $\chi_\beta \wedge \neg\beta\psi \in W$. Note that by the extended event model partial functionality axiom Proposition 3.13 (with $\neg\psi$ instantiated for φ) and both propositional and modal logic, $\vdash \neg\beta\psi \to \mathrm{P{\small RE}}(\beta)$. But then the one-point path $[W]$ is a good path from $[W]$ for $\neg\beta\Box^*_{\mathbb{B}}\psi$. Thus $[W] \notin S_\beta$, and this is a contradiction. So indeed, $\chi_\beta \wedge \neg\beta\psi$ is inconsistent. Therefore, $\vdash \chi_\beta \to \beta\psi$.

We next show that for all $A \in \mathbb{B}$ and all γ such that $\beta \xrightarrow{A} \gamma$, $\chi_\beta \wedge \mathrm{P{\small RE}}(\beta) \wedge \Diamond_A\neg\chi_\gamma$ is inconsistent. Otherwise, there would be $[W] \in S_\beta$ with χ_β, $\mathrm{P{\small RE}}(\beta)$, and $\Diamond_A\neg\chi_\gamma$ in it. Then $\bigvee\{\Diamond_A X^* : [X] \notin S_\gamma\}$, being equivalent to $\Diamond_A\neg\chi_\gamma$, would belong to W. It follows that $\Diamond_A X^* \in W$ for some $[X] \notin S_\gamma$. By Proposition 5.9, $[W] \xrightarrow{A} [X]$. Since $[X] \notin S_\gamma$, there is a good path from $[X]$ for $\neg\gamma\Box^*_{\mathbb{B}}\psi$. But since $\beta \xrightarrow{A} \gamma$ and $[W]$ contains $\mathrm{P{\small RE}}(\beta)$, we also have a good path from $[W]$ for $\neg\beta\Box^*_{\mathbb{B}}\psi$. This again contradicts $[W] \in S_\beta$. As a result, for all relevant A, β, and γ, we have that $\vdash \chi_\beta \wedge \mathrm{P{\small RE}}(\beta) \to \Box_A\chi_\gamma$.

By the event rule, $\vdash \chi_\alpha \to \alpha\Box^*_{\mathbb{B}}\psi$. Now $\vdash V_0^* \to \chi_\alpha$. So $\vdash V_0^* \to \alpha\Box^*_{\mathbb{B}}\psi$. This contradicts the assumption with which we began this proof. $\Box$

Proposition 5.13 (*Existence Lemma for Y_F*)**.** *If $U^* \wedge \widehat{Y}V^*$ is consistent, then $[U]Y_F[V]$.*

Proof. (This proof is similar to existence proof of $\xrightarrow{A}_F$ in [3]) Suppose that it is not the case that $[U]Y_F[V]$. Then there is a ψ such that $\bar{Y}\psi \in U \cap \Delta$ but $\psi \notin V$. Then $\psi \in \Delta$ and $\neg\psi \in V$, and hence $\vdash V^* \to \neg\psi$. Then $\vdash \widehat{Y}V^* \to \widehat{Y}\neg\psi$, and so $\vdash U^* \wedge \widehat{Y}V^* \to \bar{Y}\psi \wedge \widehat{Y}\neg\psi$. Hence $U^* \wedge \widehat{Y}V^*$ is inconsistent. The desired result comes from the contrapositive. $\square$

Corollary 5.14. *If $U^* \wedge \widehat{Y}$ true is consistent, then there exists a $[V] \in S_F$ such that $[U]Y_F[V]$.*

Proof. We argue using the contrapositive, and appeal to Lemma 5.13. Let us list the equivalence classes of S_F by $[V_1], \ldots, [V_n]$. The fact that $[U] = [V_i]$ for some i does not play an important role here. The following are equivalent:

1. $\vdash \neg(U^* \wedge \widehat{Y}V_i^*)$ for each i
2. $\vdash (\neg U^* \vee \bar{Y}\neg V_1^*) \wedge \cdots \wedge (\neg U^* \vee \bar{Y}\neg V_n^*)$
3. $\vdash \neg U^* \vee (\bar{Y}\neg V_1^* \wedge \cdots \wedge \bar{Y}\neg V_n^*)$
4. $\vdash \neg U^* \vee \bar{Y}\neg$ true
5. $\vdash \neg(U^* \wedge \widehat{Y}$ true$)$.

If there does not exist a $[V_i]$ such that $[U]Y_F[V_i]$, then we apply Lemma 5.13, to get condition (1) above, which states $\vdash \neg(U^* \wedge \widehat{Y}V_i^*)$ for each i, and hence we conclude condition (5), which states that $U^* \wedge \widehat{Y}$ true is inconsistent. $\square$

5.3. Truth lemma for filtration

Lemma 5.15 (*Truth Lemma for Filtration*)**.** *For each $\chi \in cl(\varphi)$ and equivalence class $[U] \in S_F$,*

$$\chi \in U \quad \text{iff} \quad \mathcal{M}_F, [U] \vDash \chi.$$

Proof. We prove this by induction on the complexity of a formula.

Inductive hypothesis: Suppose for each $\chi \in cl(\varphi)$ for which $c(\chi) \leq k$ and each equivalence class $[U] \in S_F$, $\chi \in U$ iff $\mathcal{M}_F, [U] \vDash \chi$.

Base case true and p**:** These are straightforward.

Base case e**:** Note that for e to be in $cl(\varphi)$, where $\mathsf{dep}(\varphi) \geq 1$ the following are equivalent:

1. $e \in U \cap cl(\varphi)$
2. $\varepsilon_F([U]) = e$
3. $\mathcal{M}_F, [U] \vDash e$

Case $\Box_A\psi$**:** First suppose that $\Box_A\psi \in U$. We wish to show that $\mathcal{M}_F, [U] \vDash \Box_A\psi$. Let $[V]$ be such that $[U] \xrightarrow{A} [V]$. Then by definition of $\xrightarrow{A}$, $\psi \in V$. Since $\psi < \Box_A\psi$, we apply the inductive hypothesis to get $\mathcal{M}_F, [V] \vDash \psi$. Conversely, suppose $\mathcal{M}_F, [U] \vDash \Box_A\psi$, and we wish to show that $\Box_A\psi \in U$. Suppose for a contradiction that $\Diamond_A\neg\psi \in U$. We observe that $\vdash \neg\psi \leftrightarrow \bigvee\{W^* : W$ is a maximal consistent set and $\neg\psi \in W\}$. Then we use the fact that $\Diamond_A$ distributes over disjunction to see that $U^* \wedge \Diamond_A\neg\psi$ is logically equivalent to $\bigvee\{U^* \wedge \Diamond_A V^* : \neg\psi \in V\}$. Since $U^* \wedge \Diamond_A\neg\psi$ is consistent, one of the disjuncts $U^* \wedge \Diamond_A V^*$ must be consistent. We assumed that V was such that $\neg\psi \in V$. Then by the inductive hypothesis, we see that $\mathcal{M}_F, [V] \vDash \neg\psi$. By Proposition 5.9, $[U] \xrightarrow{A}_F [V]$. We conclude that $\mathcal{M}, [U] \vDash \Diamond_A\neg\psi$, a contradiction.

Case $\Box_{\mathbb{B}}^*\psi$**:** This case is simpler than the case for $\alpha\Box_{\mathbb{B}}^*\psi$, but follows the same reasoning.

Case $\alpha\Box_{\mathbb{B}}^*\psi$. Let us first suppose that $\neg\alpha\Box_{\mathbb{B}}^*\psi \in U$ and we wish to show that $\mathcal{M}_F, [U] \vDash \neg\alpha\Box_{\mathbb{B}}^*\psi$. By Lemma 5.12, there is a good path from $[U]$ for $\neg\alpha\Box_{\mathbb{B}}^*\psi$. Let k be the length of the good path. For each $i \leq k$, $\text{Pre}(\alpha_i) \in U_i$ and $t(\text{Pre}(\alpha_i)) \in cl(\varphi)$, and by Proposition 4.7 and Lemmas 4.10 and 4.11, $c(t(\text{Pre}(\alpha_i))) < c(\alpha\Box_{\mathbb{B}}^*\psi)$. Thus by the inductive hypothesis, $\mathcal{M}, U_i \vDash t(\text{Pre}(\alpha_i))$ for each $i \leq k$. Also note that $t(\neg\alpha_k\psi) \in cl(\varphi)$, as $\alpha\Box_{\mathbb{B}}^*\psi \in cl(\varphi)$. As $\alpha \neq \lambda$ and by Proposition 4.7 and Lemma 4.10, $c(t(\neg\alpha_k\psi)) \leq c(\neg\alpha_k\psi) < c(\alpha\Box_{\mathbb{B}}^*\psi)$. Since the path is good, U_i contains $t(\neg\alpha_k\psi)$. Hence by the inductive hypothesis, $\mathcal{M}, [U_k] \vDash t(\neg\alpha_k\psi)$. Our desired result follows from the semantics. Conversely, suppose $\mathcal{M}, [U_k] \vDash \neg\alpha\Box_{\mathbb{B}}^*\psi$. By the semantics, there is a path in $\mathcal{M}$ witnessing this. A similar argument to the one used for the converse can be applied here to show that the path is good from $[U]$ for $\neg\alpha\neg_{\mathbb{B}}^*\psi$. Then by Lemma 5.11, U contains $\neg\alpha\Box_{\mathbb{B}}^*\psi$.

Case $\bar{Y}\psi$: (This argument is essentially the same as the argument for $\Box_A$.) Suppose that $\bar{Y}\psi \in U$. We wish to show that $(\mathcal{M}_F, [U]) \vDash \bar{Y}\psi$. Suppose $[V]$ is such that $[U]Y_F[V]$. Because $\bar{Y}\psi \in U \cap cl(\varphi)$, we have that $\psi \in V$. By the inductive hypothesis, $(\mathcal{M}_F, [V]) \vDash \psi$. Conversely, suppose that $(\mathcal{M}_F, [U]) \vDash \bar{Y}\psi$, and toward a contradiction that $\widehat{Y}\neg\psi \in U$. Then $U^* \cap \widehat{Y}\neg\psi$ is consistent. We observe that $\vdash \neg\psi \leftrightarrow \bigvee\{W^* : W$ is a maximal consistent set and $\neg\psi \in W\}$. Then we use the fact that $\widehat{Y}$ distributes over disjunction to see that $U^* \wedge \widehat{Y}V^*$ is logically equivalent to $\bigvee\{(U^* \wedge \widehat{Y}V^*) : \neg\psi \in V\}$. Since $U^* \wedge \widehat{Y}\neg\psi$ is consistent, one of the disjuncts $U^* \wedge \widehat{Y}V^*$ must be consistent. We assumed that V was such that $\neg\psi \in V$. Then by the inductive hypothesis $(\mathcal{M}_F, [V]) \vDash \neg\psi$. By Lemma 5.13, $[U]Y_F[V]$, and thus $(\mathcal{M}_F, [U]) \vDash \widehat{Y}\neg\psi$. $\square$

5.4. Properties of the filtration

Our aim is to construct a characterizable ETM (epistemic temporal history), which was defined by Definition 4.14). The following lemma specifies which properties of an ETH we have, and we will see that not all properties are guaranteed. The numbering in the list of properties in the lemma that follows matches the numbers for those in Definition 4.14.

Lemma 5.16. *If $\mathsf{dep}(\varphi) \geq 1$, then the filtration $\mathcal{M}_F$ has the following properties.*

1. *(Event points) $\varepsilon_F([U]) \neq \emptyset$ iff there exists $[V]$ such that $UY_F V$.*
8. *(Relation b) If $[U] \xrightarrow{A}_F [V]$ and $\varepsilon_F([U]) \neq \emptyset$, then $\varepsilon_F([U]) \xrightarrow{A}_{\mathcal{F}} \varepsilon_F([V])$.*
10. *(Valuation) If $[U]Y_F[V]$, then $\mathcal{M}_F, [U] \vDash p$ iff $\mathcal{M}_F, [V] \vDash p$.*

Proof. The proof of these come from the definition of the closure function *cl*.

Event points: This condition comes from the axioms *past event point mix* together with the fact that event points formulas e and $\widehat{Y}$ true are in $cl(\varphi)$ whenever $\mathsf{dep}(\varphi) \geq 1$, allowing us to apply the truth lemma.

Relation *b*: This condition follows from the axiom *restriction* together with the definition of *cl*.

Valuation: Suppose that $[U]Y_F[V]$ and $[U] \in \|p\|_F$. Then $p \in U \cap cl(\varphi)$, and for that to happen, p must occur in φ. If $\mathsf{dep}(\varphi) \geq 1$, then $\bar{Y}p \in cl(\varphi)$, and by consistency, $\bar{Y}p \in U$. By the truth lemma $\mathcal{M}_F, [U] \vDash \bar{Y}p$, and thus $\mathcal{M}_F, [V] \vDash p$, whence $[V] \in \|p\|_F$.

Conversely, suppose that for every $[V]$ for which $[U]Y[V]$, $[V] \in \|p\|_F$. Suppose also that there exists a $[V]$ such that $[U]Y_F[V]$. Then $p \in V \cap cl(\varphi)$, and hence $\mathcal{M}, [V] \vDash p$. As this is true for every such $[V]$, we know that $\mathcal{M}, [U] \vDash \bar{Y}p$. As long as $\mathsf{dep}(\varphi) \geq 1$, we have $\bar{Y}p \in cl(\varphi)$ by the definition of *cl*. Hence by the truth lemma $\bar{Y}p \in U \cap cl(\varphi)$. Similarly, we can establish that $\widehat{Y}$ true $\in U \cap cl(\varphi)$, and hence by consistency $p \in U \cap cl(\varphi)$ (that is $[U] \in \|p\|_F$). □

Thus we can only guarantee three of the ten properties that an ETM must have in order to be an ETH. What about the other seven properties? Four of the seven do not correspond to the proof system at all. These are 3 (*bounded age*), 5 (*states a*), 6 (*states b*), and 9 (*relation c*). The other three of the seven properties that we lack are 2 (*partial functionality of Y*), 4 (*synchronicity*), and 7 (*relation a*). Note that these correspond to infinite axiom schema, where an infinite axiom scheme has infinitely many instantiations. As $cl(\varphi)$ is finite, the filtration can only accommodate a finite number of axioms out of an infinite axiom scheme. In some cases, only finitely many instantiations of an infinite axiom scheme are needed. This would be the case with *past atomic permanence*, which would be infinite if the set Φ of atomic propositions were infinite. Only the axioms corresponding to proposition letters in the given consistent formula φ are needed. But with *partial functionality of Y*, *synchronicity*, and *relation a*, finitely many instantiations does not seem to be enough to guarantee the corresponding properties in the filtration.

We can, however, establish synchronicity up to the yesterday depth of the formula φ, and it turns out that we will be able to transform our filtration into an equivalent one (equivalent in the eyes of the formula φ) that is trimmed at this yesterday depth, thus ensuring full synchronicity.

Proposition 5.17 (*Bounded Synchronization*)**.** *If $\mathsf{dep}(\varphi) \geq 1$ and $[U] \xrightarrow{A} [V]$, then if $k \leq \mathsf{dep}(\varphi)$, $[U]Y^k[U']$ for some $[U']$ if and only if $[V]Y^k[V']$ for some $[V']$.*

Proof. We assume that $\mathsf{dep}(\varphi) \geq 1$ and that $[U] \xrightarrow{A} [V]$. Then for the first direction of the biconditional, assume that $[U]Y^k[U']$. By Definition 5.3 (items 3 and 6), $\neg\bar{Y}^k$ false and $\Box_A\neg\bar{Y}^k$ false are in $cl(\varphi)$, and as $\mathcal{M}, [U] \vDash \neg\bar{Y}^k$ false, we use the truth lemma to get $\neg\bar{Y}^k$ false $\in U$. By consistency, $\Box_A\neg\bar{Y}^k$ false $\in U$. Finally, by the truth lemma $\mathcal{M}, [U] \vDash \Box_A\neg\bar{Y}^k$ false, from which our desired result is immediate.

For the other direction, we assume that $[U]Y^k[U']$ for no state $[U']$, and the rest of the argument is similar. □

5.5. General strategy

The general strategy for the rest of the proof is to shape the filtration into another ETM that does have all 10 desirable properties. To establish these, we will perform the following types of transformations.

Unraveling: The first transformation will be to unravel the filtration into a bisimilar ETM that is more tree-like in structure. A tree-like structure allows a state to be related to many different states, but no two states can be related to the same state. This would make conditions 5 (*states a*) and 9 (*relation c*) hold vacuously, as the antecedents in these conditions involve two states related to the same state. In order to establish decidability, we will aim for a finite ETH at the end of all transformation. The particular unraveling we will then use will not produce a tree, where no two states are related to the same, but will produce a model that is enough like a tree to ensure that no two states can be related to the same state if one of the relational connections is Y. This will still ensure conditions 5 and 9 hold vacuously.

Trimming: There are two ways in which we will trim the unraveled model. One is to remove any state whose path from the root of the (partially) tree-like unraveled structure contains no more Y-relational steps than $\mathtt{dep}(\varphi)$, where φ is the fixed consistent formula we are finding a model for. This trimming will ensure we have conditions 3 (*bounded age*) and less obviously 4 (*epistemic synchronicity*), but we will lose condition 1 (*event points*), as there may be states that are not Y-related to any other state, but assigned an event point by ε. The way to resolve this will be to modify the ε function slightly, and have the truth lemma only apply to formulas in $cl_k(\varphi)$ at states that are k Y-relational steps from such a Y-terminal state.

The other type of trimming will be to select from every state one Y-related state to keep and remove all other Y-related states. This will ensure condition 2 (*partial functionality of Y*).

Modifying ε: This step will redefine the ε function at states that are not Y-related to any other state, such that these states are mapped to $\emptyset$. Note that the $\emptyset$ indicates just that fact that the state mapped to it is not Y-related to any other state. This modification will reestablish condition 1 (*event points*).

Expanding $\xrightarrow{A}$: The whole goal of this step is to establish condition 7 (*relation a*), also known as perfect recall. It asserts that if there is a relational connection between two states (indicating uncertainty), then there was a corresponding relational connection at the previous stage (given by the states' Y-successors). The modification thus adds epistemic relational connections between past states when needed. Now introducing new relational connections may disrupt the property we established from unraveling, that guarantees that no two states are related to the same state if one of the relational connections is a Y connection. Recall that the two conditions we were able to obtain from this are 5 and 9. Condition 5 is vacuously true as long as no two states are Y-related to the same state, and this remains true as we are only adding epistemic relational connections. Condition 9 states that if there was a relational connection before, their future states will also be related to each other, unless the event points corresponding to them indicate there was learning. Since the unraveling step, no two states are Y-related to the same state; hence each state has a unique future. Thus for a relational connection to be introduced during this step, there must already have been a relational connection between the corresponding future states, and hence condition 9 will still hold as well.

Modifying $\|\cdot\|$: This last step is only needed if we wish to ensure that the resulting ETH is characterizable. Its entire goal is to ensure that condition 6 (*states b*) holds, which asserts that the set of states at one stage that can be successfully updated is characterizable by a formula ψ; that is, the set of states at a particular distance from a Y-terminal state in which ψ is true are those states with a Y-predecessor. An easy way to do this that does not interfere with any of the other properties that have already been established, is to make such formulas atomic propositions that do not occur in the formula φ which we are trying to find a satisfiable ETH for. We thus need an arbitrary number of proposition letters available. Thus completeness with respect to characterizable histories requires the set of symbols of the language be infinite (with infinitely many atomic propositions), and completeness with respect to histories that are not necessarily characterizable can be established for languages with finitely many symbols.

We depict this discussion in the following table. The numbers 1 through 10 correspond to the 10 properties of an ETH (Definition 4.14).

	1	2	3	4	5	6	7	8	9	10
filtration	Y	-	-	-	-	-	-	Y	-	Y
unraveling	Y	-	-	-	Y	-	-	Y	Y	Y
trimming	-	Y	Y	Y	Y	-	-	Y	Y	Y
modifying ε	Y	Y	Y	Y	Y	-	-	Y	Y	Y
expanding $\xrightarrow{A}$	Y	Y	Y	Y	Y	-	Y	Y	Y	Y
modifying $\|\cdot\|$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

Y means that the model has the property, and '-' means that the property is not guaranteed.

To be more concise in the sections that follow, we will combine the step *trimming* with *modifying* ε and *expanding* $\xrightarrow{A}$ with *modifying* $\|\cdot\|$. It is not as easy to combine any of the unraveling, trimming, or expanding steps together, and it is helpful to focus on them separately.

5.6. Unraveling

Unraveling is a method of transforming a model into another whose states contain a record of the relational steps taken to get from a particular given state. As we have discussed in the last section, unraveling will be key to establishing conditions 5 (*states a*) and 9 (*relation b*). The most common type of unraveling is to record every relational step of every type of relation from a particular state, and the result of this unraveling will be a tree, which is likely to be infinite. As one of the operations we will be performing on our model is to trim away the states that have too many Y-relational connections from the root, we will unravel in the Y direction, but maintain the finiteness of the epistemic relational structure. Thus we will use the following partial unraveling.

Let φ be our fixed formula in $\mathcal{L}_{\mathsf{rf}}$ for which we are trying to find a model. Let $[U]$ be an equivalence class containing φ, whence $\mathcal{M}_F, [U] \vDash \varphi$. We then generate the following new set of states.

Definition 5.18 (*Set S_U*)**.** Let S_U be the smallest set containing $[U]$ that is closed under the following operations:

1. $[U] \in S_U$
2. $z\mathrm{E}[W] \in S_U$ if $z \in S_U$ and either
   (a) $z = [U]$ and $[U] \xrightarrow{A}_F [W]$ for some $A \in \mathbb{A}$.
   (b) $z\mathrm{E}[V] \in S_U$ for some $[V] \in S_F$ in which $[V] \xrightarrow{A}_F [W]$ for some $A \in \mathbb{A}$.
   (c) $z = x\mathrm{Y}[V]$, with $x \in S_U$ and $[V] \xrightarrow{A}_F [W]$ for some $A \in \mathbb{A}$
3. $z\mathrm{Y}[W] \in S_U$ if $z \in S_U$ and either
   (a) $z = [U]$ and $[U]Y_F[V]$
   (b) $z = x\mathrm{E}[V]$, with $x \in S_U$ and $[V]Y_F[W]$
   (c) $z = x\mathrm{Y}[V]$, with $x \in S_U$ and $[V]Y_F[W]$.

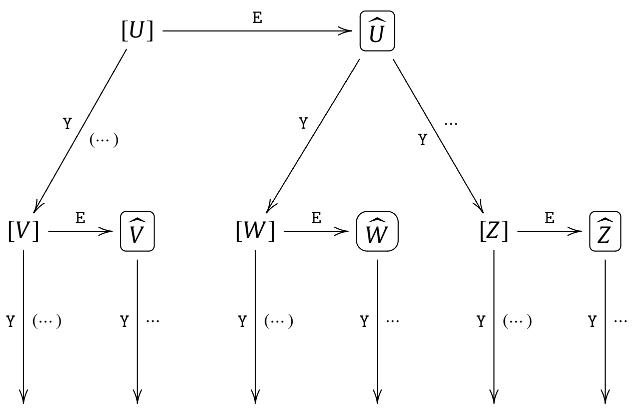
Note that part 2 (b) is different from the other parts, in that it does not consider the structure of z. Considering the structure of z would have been a natural way of defining states such as $[U]\mathrm{E}[V]\mathrm{E}[W]$, where there is an S_F state directly between two E's. But this would be too similar to a complete unraveling, and we wish to avoid such situations. The intention of this definition rather is to indicate that $[U]\mathrm{E}[V] \in S_U$ only if $[U] \xrightarrow{\mathbb{A}}{}^{*}_F [V]$ rather than $[U] \xrightarrow{A}_F [V]$ for some $A \in \mathbb{A}$.

We define a function $\mathsf{cls} : S_U \to S_F$ by $\mathsf{cls}([V]) = [V]$ for each $[V] \in S_U \cap S_F$, and $\mathsf{cls}(x\mathrm{E}[V]) = \mathsf{cls}(x\mathrm{Y}[V]) = [V]$. We then generate the partial unraveling as the following structure.

Definition 5.19 (*Unraveled Filtration*)**.** Define $\mathcal{M}_U = (S_U, \to_U, Y_U, \varepsilon_U, \|\cdot\|_U)$ by

1. S_U is defined by Definition 5.18
2. $x \xrightarrow{A}_U z$ iff $\mathsf{cls}(x) \xrightarrow{A}_F \mathsf{cls}(z)$ and any one of the following conditions hold:
   (a) $x = [U]$ and $z = [U]\mathrm{E}[V]$ for some $[V] \in S_F$
   (b) $x = w\mathrm{E}[V_1]$ and $z = w\mathrm{E}[V_2]$ where $w \in S_U$ and $[V_1], [V_2] \in S_F$
   (c) $x = w\mathrm{Y}[V_1]$ and $z = x\mathrm{E}[V_2]$ for $w \in S_U$ and $[V_1], [V_2] \in S_F$
3. xY_Uz iff $z = x\mathrm{Y}\,\mathsf{cls}(z)$ and $\mathsf{cls}(x)Y_F\,\mathsf{cls}(z)$
4. $\varepsilon_U(x) = \varepsilon_F(\mathsf{cls}(x))$
5. $x \in \|p\|_U$ iff $\mathsf{cls}(x) \in \|p\|_F$.

We try to visualize the relational structure of $\mathcal{M}_U$ as follows:



where for each V, $\widehat{V}$ is (isomorphic to) the submodel of $\mathcal{M}_F$ consisting of states epistemically reachable from $[V]$. The E above the arrows is suggestive of the names of the states, and each $\xrightarrow{E}$ corresponds to possibly many relational connections for each agent. The symbol Y on a vertical branch represents one Y-relational connection. It is conceivable that there may be an infinite string of these Y-relational connections. We have only shown strings of length two represented by two vertical branches in series. One goal of Section 5.7 will be to ensure that such consecutive branches are no more than $\mathsf{dep}(\varphi)$ in length. The dots . . . next to the Y on some vertical branches tell us that the branch need not be the last one, that there may be many more vertical branches. Due to Proposition 5.17, each state in the submodel, say $\widehat{V}$, will have at least one branch coming out of it, assuming that $\mathsf{dep}(\varphi)$ is greater than the number of Y steps above it. The parenthetical dots (. . .) emphasize the undesirable possibility that there may be more than one Y branch from the same state. Another goal of Section 5.7 will be to eliminate such a possibility, and to ensure that the relation Y is a partial function.

5.6.1. Truth lemma for $\mathcal{M}_U$

We shall use the following bisimulation to establish a truth lemma for $\mathcal{M}_U$.

Lemma 5.20 (*Bisimulation between Filtration and Unraveled Filtration*)**.** *The function* cls *from S_U to S_F defines a bisimulation between $\mathcal{M}_U$ and $\mathcal{M}_F$.*

Proof. By definition of $\mathcal{M}_U$, for each $x \in S_U$ and $p \in \Phi$, $x \in \|p\|_U$ iff $\mathsf{cls}(x) \in \|p\|_F$. Also by definition $\varepsilon_U(x) = \varepsilon_F(\mathsf{cls}(x))$.

Next, note that it is required that $\mathsf{cls}(x) \xrightarrow{A}_F \mathsf{cls}(z)$ for $x \xrightarrow{A}_U z$. Suppose $\mathsf{cls}(x) \xrightarrow{A}_F [W]$. If $x = [U]$, let $z = [U]\mathrm{E}[W]$; if $x = a\mathrm{E}[V]$, let $z = a\mathrm{E}[W]$; if $x = a\mathrm{Y}[V]$, let $z = x\mathrm{E}[W]$. There are no other forms that x can take, and in each of these cases, $x \xrightarrow{A}_U z$ and $\mathsf{cls}(z) = [W]$.

Next, note that it is required that $\mathsf{cls}(x) Y_F \mathsf{cls}(z)$ for xY_Uz. Suppose that $\mathsf{cls}(x) Y_F [W]$. Set $z = x\mathrm{Y}[W]$. Then xY_Uz and $\mathsf{cls}(x) Y_F \mathsf{cls}(z)$. $\square$

Define fml by $\mathsf{fml}(x) = V \cap cl(\varphi)$, for some $V \in \mathsf{cls}(x)$ (note that we are defining fml for a fixed φ).

Lemma 5.21 (*Truth Lemma for Unraveled Filtration*)**.** *For each $\chi \in cl(\varphi)$ and $x \in S_U$,*

$$\chi \in \mathit{fml}(x) \quad \mathit{iff} \quad \mathcal{M}_U, x \vDash \chi.$$

Proof. By Lemmas 5.20 and 4.20, given any $\psi \in \mathcal{L}_{\mathrm{rf}}$ and $x \in S_U$, $\mathcal{M}_F, \mathsf{cls}(x) \vDash \psi$ iff $\mathcal{M}_U, x \vDash \psi$. We appeal to the truth lemma for the filtration $\mathcal{M}_F$ to obtain the desired result. $\square$

5.6.2. Properties of $\mathcal{M}_U$

Lemma 5.22. *If $\mathit{dep}(\varphi) \geq 1$, then the unraveled model $\mathcal{M}_U$ has properties* 1 *(event points),* 8 *(relation b), and* 10 *(valuation) (which are the properties that $\mathcal{M}_F$ has) together with the following.*

5. *(States a) If uYz, xYz, and $u \neq x$, then $\varepsilon(u) \neq \varepsilon(x)$.*
9. *(Relation c) If xYx', zYz', $x' \xrightarrow{A} z'$, and $\varepsilon(x) \xrightarrow{A} \varepsilon(z)$, then $x \xrightarrow{A} z$.*

Proof. 1. Event points: This condition states that $\varepsilon(x) \neq \emptyset$ iff there exists a y such that xY_Uy. Then the following are equivalent.

1. $\varepsilon_U(x) \neq \emptyset$.
2. $\varepsilon_F(\mathsf{cls}(x)) \neq \emptyset$.
3. There is a $[Z]$ such that $\mathsf{cls}(x) Y_F [Z]$.
4. There is a z such that xY_Uz.

The equivalence of (2) and (3) is from Lemma 5.16. The equivalence of (3) and (4) is immediate by setting $z = x\mathrm{Y}[Z]$.

5. States a: This property is vacuously true, as there can be no two states Y related to the same state. For suppose uYz and xYz. Then by definition of $\mathcal{M}_U$, $z = u\mathrm{Y}\,\mathsf{cls}(z)$ and $z = x\mathrm{Y}\,\mathsf{cls}(z)$. Hence $u = x$.

8. Relation b: This property states that if $x \xrightarrow{A}_U z$ and $\varepsilon_U(x) \neq \emptyset$, then $\varepsilon_U(x) \xrightarrow{A}_{\mathcal{F}} \varepsilon_U(z)$. Here we use the facts from the definition of $\mathcal{M}_U$ that $x \xrightarrow{A}_U z$ implies $\mathsf{cls}(x) \xrightarrow{A}_F \mathsf{cls}(z)$ and that $\varepsilon_U(x) = \varepsilon_F(\mathsf{cls}(x))$, and the desired result then follows from Lemma 5.16.

9. Relation c: This property is vacuously true, as no two states in $\mathcal{M}_U$ are related to the same state, when one of the relations is Y. Suppose that uYz and $x \xrightarrow{A} z$. Then $z = u\mathrm{Y}\,\mathsf{cls}(z)$ from the fact that uYz. But $x \xrightarrow{A} z$ requires that $z = x\mathrm{E}\,\mathsf{cls}(z)$ contradicting that $z = u\mathrm{Y}\,\mathsf{cls}(z)$.

10. Valuation: This property states that if xY_Uz, then $\mathcal{M}_U, x \vDash$ iff $\mathcal{M}_U, z \vDash$. Then assuming that xY_Uz, we see by definition that $\mathsf{cls}(x) Y_F \mathsf{cls}(z)$, whence we can use Lemma 5.16, to obtain that $\mathcal{M}_F, \mathsf{cls}(x) \vDash p$ iff $\mathcal{M}_F, \mathsf{cls}(z) \vDash p$, which also means that $\mathsf{cls}(x) \in \|p\|_F$ iff $\mathsf{cls}(z) \in \|p\|_F$. Our desired result is immediate from the definition of $\|p\|_U$. $\square$

5.7. Trimming the unraveled filtration

For each state $x \in S_U$, we will denote by $\mathsf{rootd}(x)$ the number of Y_U-relational steps from $[U]$ to x. More formally we have the following.

Definition 5.23 (*Root Yesterday Distance*)**.** We define the function $\mathsf{rootd} : S_U \to \mathbb{N}$ as follows:

$$\mathsf{rootd}(x) = \begin{cases} 0 & \text{if } x = [U] \\ \mathsf{rootd}(z) & \text{if } x = z\mathrm{E}[V] \\ \mathsf{rootd}(z) + 1 & \text{if } x = z\mathrm{Y}[V]. \end{cases}$$

We will now trim our model by removing states from S_U. Our aim is to bound the root yesterday distance to the modal depth of φ, and also to ensure that the relation Y is a partial function.

Definition 5.24 (*Set S_T*)**.** Let us define the set S_T as the smallest set for which

1. $[U] \in S_T$
2. if $x \in S_T$ and $x \xrightarrow{A}_U y$, then $y \in S_T$
3. if $x \in S_T$, $\mathsf{rootd}(x) < \mathsf{dep}(\varphi)$, and $Z = \{y : xY_U y\}$, then exactly one $y \in Z$ shall be in S_T. The element in Z to be selected is arbitrary.

Note that S_T does not contain any state x where $\mathsf{rootd}(x) > \mathsf{dep}(\varphi)$. Then we define a new structure as follows:

Definition 5.25 (*Trimmed Unraveled Filtration*)**.** Let $\mathcal{M}_T = (S_T, \{\xrightarrow{A}_T\}, Y_T, \varepsilon_T, \|\cdot\|_T)$ be defined by

1. S_T is defined by Definition 5.24,
2. $\xrightarrow{A}_T = (\xrightarrow{A}_U \cap S_T^2)$ is the restriction of $\xrightarrow{A}_U$ to S_T,
3. $Y_T = Y_U \cap S_T^2$ is the restriction of Y_U to S_T,
4. $\varepsilon_T : S_T \to E \cup \{\emptyset\}$ is defined by $\varepsilon_T(x) = \begin{cases} \varepsilon_U(x) & \text{if } \mathsf{rootd}(x) < \mathsf{dep}(\varphi) \\ \emptyset & \text{if } \mathsf{rootd}(x) = \mathsf{dep}(\varphi), \end{cases}$
5. $\|\cdot\|_T$ is given by $\|p\|_T = \|p\|_U \cap S_T$.

5.7.1. Truth lemma for $\mathcal{M}_T$

Define the leaf yesterday distance of a state x, written $\mathsf{leafd}(x)$ to be the number of Y_T steps from x to a terminal state. Note that $\mathsf{leafd}(x) + \mathsf{rootd}(x) + K = \mathsf{dep}(\varphi)$, where K is non-zero (and positive) in the event that there is no state x in S_U for which $\mathsf{rootd}(x) \geq \mathsf{dep}(\varphi)$. In such a situation where K is positive, we would have no need for the layered sets $cl_j(\varphi)$ from Definition 5.5, and the overall proof ahead can be simplified. But let us assume for the rest of the completeness proof the harder case where K is zero.

Lemma 5.26 (*Layered Truth Lemma for $\mathcal{M}_T$*)**.** *For each j for which $0 \leq j \leq dep(\varphi)$, each $\chi \in cl_j(\varphi)$, and each $x \in S_T$, for which* $\mathsf{leafd}(x) = j$,

$$\chi \in fml(x) \quad \textit{iff} \quad \mathcal{M}_T, x \vDash \chi.$$

Proof. We prove this by showing that for each $x \in S_T$ with leaf yesterday distance j, and each $\chi \in cl_j(\varphi)$, $\mathcal{M}_T, x \vDash \chi$ iff $\mathcal{M}_U, x \vDash \chi$. Our desired result follows from the truth lemma for $\mathcal{M}_U$. We then prove this using induction on j, and for each j, an internal induction on the complexity of the formula.

Outer inductive hypothesis: For each $k < j$, if $\chi \in cl_k(\varphi)$ and $x \in S_T$ with $\mathsf{leafd}(x) = k$, then $\mathcal{M}_T, x \vDash \chi$ iff $\mathcal{M}_U, x \vDash \chi$.

Inner inductive hypothesis: Whenever $c(\psi) < c(\chi)$, if $\psi \in cl_k(\varphi)$ and $x \in S_T$ with $\mathsf{leafd}(x) = k$, then $\mathcal{M}_T, x \vDash \psi$ iff $\mathcal{M}_U, x \vDash \psi$.

Base cases true and p**:** These internal induction base cases use the same reasoning regardless of the value of j. The case for true is trivial, and the case for p comes from the fact that the function $\|\cdot\|$ in both models treats each state in S_T the same way.

Base case e: Note that as $\mathsf{dep}(e) = 1 > 0$, it is the case that $e \notin cl_0(\varphi)$. Thus this case only arises when $j > 0$, and for each j we use the fact that the function ε treats each $x \in S_T$ for which $\mathsf{leafd}(x) > 0$ the same in both models.

Cases $\neg$ and $\wedge$**:** The Boolean cases are trivial.

Case $\bar{Y}\psi$: This is the one inductive step that makes use of the outer inductive hypothesis. Note that as $\mathsf{dep}(\bar{Y}\psi) > 0$, it is the case that $\bar{Y}\psi \notin cl_0(\varphi)$, and hence this case only arises when $j > 0$. Suppose that for $\bar{Y}\psi \in cl_j(\varphi)$ and $x \in S_T$ with $\mathsf{leafd}(x) = j$, $\mathcal{M}_T, x \nvDash \bar{Y}\psi$. Then there exists z such that xY_Tz and $\mathcal{M}_T, z \nvDash \psi$. As $S_T \subseteq S_U$, we have $z \in S_U$. As $\bar{Y}\psi$ is in $cl(\varphi)$, so is ψ, and as $\mathsf{dep}(\bar{Y}\psi) \leq j$, $\mathsf{dep}(\psi) \leq j - 1$, whence $\psi \in cl_{j-1}(\varphi)$. This allows us to apply the inductive hypothesis to obtain $\mathcal{M}_U, z \nvDash \psi$. Our desired result follows from the fact that $x \in S_U$ and xY_Uz.

The converse is similar, but with an added complication. We suppose that for $\bar{Y}\psi \in cl_j(\varphi)$ and $x \in S_T$ with $\mathsf{leafd}(x) = j$, $\mathcal{M}_U, x \nvDash \bar{Y}\psi$, and obtain a z such that xY_Uz and $\mathcal{M}_U, z \nvDash \psi$. We do not have a guarantee that $z \in S_T$. But recall that the truth lemma of $\mathcal{M}_U$ applies, whence $\psi \in \mathsf{fml}(x)$. Note that from the definition of cl, as $\bar{Y}\psi \in cl_j(\varphi)$, so also $\neg\bar{Y}\psi, \bar{Y}\sim\psi \in cl_j(\varphi)$. Hence $\neg\bar{Y}\psi \in \mathsf{fml}(x)$, and by the axiom *non-branching past* and propositional logic, $\bar{Y}\sim\psi \in \mathsf{fml}(x)$. We apply the truth lemma again to arrive at $\mathcal{M}_U, x \vDash \bar{Y}\sim\psi$. If z' is such that xY_Tz', then $\mathcal{M}_U, z' \vDash \sim\psi$. The inductive hypothesis applies in the same way as it did for the converse.

Cases $\Box_A\psi$ and $\Box^*_{\mathbb{B}}\psi$**:** These steps use the fact that for each $x \in \mathcal{M}_T$, $\{z : x \xrightarrow{A}_T z\} = \{z : x \xrightarrow{A}_U z\}$.

Case $\alpha\Box^*_{\mathbb{B}}\psi$**:** Given a state x in S_T with leaf yesterday distance j (for $0 \leq j \leq \mathsf{dep}(\varphi)$), let us assume $\mathcal{M}_T, x \nvDash \alpha\Box^*_{\mathbb{B}}\psi$. Then by the semantics of ETMs (Definition 4.13), there exist sequences

$$x = x_0 \xrightarrow{A_1} x_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} x_{k-1} \xrightarrow{A_k} x_k$$

and

$$\alpha = \alpha_0 \xrightarrow{A_1} \alpha_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{k-1}} \alpha_{k-1} \xrightarrow{A_k} \alpha_k$$

with each $A_i \in \mathbb{B}$, $\mathcal{M}_T, x_i \vDash t(\textsc{Pre}(\alpha_i))$ for each $i \leq k$, and $\mathcal{M}_T, x_k \vDash t(\neg\alpha_k\psi)$. Note that each state in the path has the same leaf (or root) yesterday depth. Also note that the set of formulas $cl_j(\varphi)$ also contains $t(\textsc{Pre}(\alpha_i))$ for each i, as well as $t(\alpha_i\psi)$ from the definition of cl and the fact that the yesterday depth of the formulas are at most j. Also note that $c(t(\textsc{Pre}(\alpha_i))) \leq c(\textsc{Pre}(\alpha_i)) < c(\alpha_i\psi) < c(\alpha\Box_{\mathbb{B}}^*\psi)$ and $c(t(\alpha_i\psi)) \leq c(\alpha_i\psi) < c(\alpha\Box_{\mathbb{B}}^*\psi)$ by Proposition 4.7 and Lemmas 4.10 and 4.11. Thus we may use the inner inductive hypothesis for each formula involved in the semantics of $\alpha\Box_{\mathbb{B}}^*\psi$ to obtain $\mathcal{M}_U, x \nvDash \alpha\Box_{\mathbb{B}}^*\psi$. To attain the converse, note that the converse of each step is true, since for each $x \in \mathcal{M}_T$, $\{z : x \xrightarrow{A}_T z\} = \{z : x \xrightarrow{A}_U z\}$. $\Box$

5.7.2. Properties of $\mathcal{M}_T$

Lemma 5.27. *$\mathcal{M}_T$ satisfies all properties of an epistemic temporal history, given by Definition 4.14, except for 6 (states b) and 7 (relation a).*

Proof. Properties 1, 5, 8, 9, and 10 are preserved from the unraveled model. The trimming of the model alone could make property 1 fail at states of leaf yesterday distance 0, but the definition of ε_T makes sure such Y-terminal states are mapped to $\emptyset$, as they should. The other properties are not affected by this trimming. This is because xY_Uz iff there exists $z' \in S_T$ such that xY_Tz' (and xY_Uz' as well), and for every $x \in S_T$ and every $z \in S_U$, it is the case that $x \xrightarrow{A}_T z$ iff $x \xrightarrow{A}_U z$. But now $\mathcal{M}_T$ also satisfies condition 2 (*partial functionality of Y*), as Y_T has been defined to be a partial function. Property 3 (*bounded age*) holds in $\mathcal{M}_T$, as its trimming bounds the age to be $\mathsf{dst}(\varphi)$. Property 4 (*synchronicity*) comes from Lemma 5.17 and the bisimulation from Lemma 5.20. $\Box$

5.8. Establishing a history

We now want to establish properties 6 (*states b*) and 7 (*relation a*) of an epistemic temporal history, the only two conditions not satisfied by $\mathcal{M}_T$. Recall that *states b* asserts that the history is characterizable, and *relation a* ensures that agents do not forget permanent facts.

As $\mathcal{M}_T$ already satisfies both condition 1 (*event points*) and condition 2 (*partial functionality of Y*), we can identify each element x for which xY_Tz with a unique pair (z, e). The condition *event points* ensures that there will be such an event point e (it is unique, as ε_T is a function). The condition *partial functionality of Y* ensures that such a z is unique.

To establish property 6 (*states b*), let m be the number of event points in E, the carrier set of the event frame, and let $n = \mathsf{dep}(\varphi)$. We label mn atomic propositions $\{q_e^k : 1 \leq k \leq n, e \in E\}$ not appearing in the formula φ. For each q_e^k, we think of k as indicating the leaf yesterday depth of a state. We then define sets Q_e^k to be the smallest subsets of S_T for which the following hold.

1. $\{x : \mathsf{leafd}(x) = k, \varepsilon_T(x) = e\} \subseteq Q_e^k$.
2. if $x \in Q_e^k$ and either xY_Tz or zY_Tx, then $z \in Q_e^k$.

Definition 5.28 (*Sequence $\mathcal{M}_i$ and $\mathcal{M}_H$*). Let m be the number of event points in E, and let $n = \mathsf{dep}(\varphi)$. We define a sequence of epistemic temporal models $\mathcal{M}_i = (S_i, \to_i, Y_i, \varepsilon_i, \|\cdot\|_i)$ for each $0 \leq i \leq n$. We set $\mathcal{M}_0 = \mathcal{M}_T$. For each i from 0 to $n-1$, we define $\mathcal{M}_{i+1}$ to be identical to $\mathcal{M}_i$ in every component except for the epistemic relations $\xrightarrow{A}$ and the valuation $\|\cdot\|$. For each agent A, we first define,

$$\xrightarrow{A}_{i+1} \equiv \xrightarrow{A}_i \cup \{(x, z) : \exists e, f \in E \text{ with } \mathsf{rootd}((x, e)) = i \text{ and } (x, e) \xrightarrow{A}_i (z, f)\}.$$

We define the valuation function to be the same in each ETM other than the first (note that the definition does not depend on i):

$$\|p\|_{i+1} = \begin{cases} \|p\|_T & p \neq q_e^k \text{ with } e \in E \text{ and } k = 1, \ldots, n \\ Q_e^k & p = q_e^k. \end{cases}$$

We define $\mathcal{M}_H = (S_H, \to_H, Y_H, \varepsilon_H, \|\cdot\|_H) = (S_n, \to_n, Y_n, \varepsilon_n, \|\cdot\|_n)$.

Truth lemma for $\mathcal{M}_H$

A key ingredient to the proof of the truth lemma is the following lemma.

Lemma 5.29. *Suppose that for each $x \in S_i$, and all $\psi \in cl_{\mathsf{leafd}(x)}(\varphi)$, $\mathcal{M}_i, x \vDash \psi$ iff $\psi \in cls(x)$, that is $\mathcal{M}_i$ satisfies the same kind of truth lemma as $\mathcal{M}_T$. Suppose also that $x \xrightarrow{A}_{i+1} z$. Then if $\mathcal{M}_i, x \vDash \Box_A\psi$, where $\Box_A\psi \in cl_{\mathsf{leafd}(x)}(\varphi)$, we then have $\mathcal{M}_i, z \vDash \psi$.*

Proof. Suppose that $\mathcal{M}_i, x \vDash \Box_A\psi$. If $x \xrightarrow{A}_i z$, then the result immediately follows from the semantics of $\Box_A$. If it is not the case that $x \xrightarrow{A}_i z$, then since $x \xrightarrow{A}_{i+1} z$, there must be event points $e, f \in E$, such that $(x, e) \xrightarrow{A}_i (z, f)$. Note that $\mathcal{M}_i, (x, e) \vDash \bar{Y}\Box_A\psi$. By the definition of cl, we have $\Box_A\bar{Y}\psi, \bar{Y}\Box_A\psi \in X_{\mathsf{leafd}(x)+1}$. Then by the axiom *epistemic past mix* and our assumption that the truth lemma applies to $\mathcal{M}_i$, we have $\mathcal{M}_i, (x, e) \vDash \Box_A\bar{Y}\psi$. Thus $\mathcal{M}_i, z \vDash \psi$. $\Box$

Lemma 5.30 (*Layered Truth Lemma for $\mathcal{M}_i$*). *For each i, j for which $0 \le i, j \le dep(\varphi)$, each $\chi \in cl_j(\varphi)$, and each $x \in S_i$, for which* $\mathsf{leafd}(x) = j$,

$$\chi \in fml(x) \quad \textit{iff} \quad \mathcal{M}_i, x \vDash \chi.$$

Proof. We show this by induction on i, the index of the ETM. We can view our base case as $i = 0$, which states that the truth lemma holds for $\mathcal{M}_T$. Assume the truth lemma holds for $\mathcal{M}_i$ ($i < \mathsf{dep}(\varphi)$). We wish to show it holds for $\mathcal{M}_{i+1}$. We then use a double induction on the leaf yesterday distance j of the state as the outer induction, and the complexity of formulas as the internal induction.

Outer inductive hypothesis: For each $k < j$, if $\chi \in cl_k(\varphi)$ and $x \in S_{i+1}$ with $\mathsf{leafd}(x) = k$, then $\mathcal{M}_{i+1}, x \vDash \chi$ iff $\mathcal{M}_i, x \vDash \chi$.

Inner inductive hypothesis: Whenever $c(\chi) < c(\psi)$, if $\chi \in cl_k(\varphi)$ and $x \in S_{i+1}$ with $\mathsf{leafd}(x) = k$, then $\mathcal{M}_{i+1}, x \vDash \chi$ iff $\mathcal{M}_i, x \vDash \chi$.

Base cases true, e, p**:** The base case true is trivial. The case for event points e holds from the fact that the function ε_i is the same for each i. Finally, the case for p holds, since for each $p \in cl(\varphi)$, we have that $\|p\|_i$ is defined the same for each i.

Cases $\neg$ and $\wedge$**:** The Boolean steps are easy.

Case $\bar{Y}\chi$: This is the only case that uses the outer induction hypothesis, and is straightforward given the fact that the Y-relation is the same for each i.

Case $\Box_A\chi$: Suppose $\psi = \Box_A\chi$, $\mathsf{leafd}(x) = j$, and $\mathcal{M}_i, x \vDash \Box_A\chi$. Suppose that $x \xrightarrow{A}_{i+1} z$. Then as $\psi \in cl_j(\varphi)$, we have by Lemma 5.29 that $\mathcal{M}_i, z \vDash \chi$. Since $c(\chi) < c(\psi)$, we can apply the inner induction hypothesis to obtain $\mathcal{M}_{i+1}, z \vDash \chi$. As z was chosen arbitrarily, $\mathcal{M}_{i+1}, x \vDash \Box_A\chi$. The converse of this $\Box_A$ case is easier to establish, and uses the fact that $\xrightarrow{A}_i$ is a subset of $\xrightarrow{A}_{i+1}$.

Case $\Box^*_{\mathbb{B}}\chi$: Suppose that $\psi = \Box^*_{\mathbb{B}}\chi$, $\mathsf{leafd}(x) = j$, and $\mathcal{M}_i, x \vDash \Box^*_{\mathbb{B}}\chi$. As $\mathcal{M}_i$ satisfies the truth lemma and as for each $A \in \mathbb{B}$, we have that $\Box_A\Box^*_{\mathbb{B}}\chi \in cl_{\mathsf{leafd}(x)}(\varphi)$ from the definition of cl, we can apply axiom *epistemic mix* to obtain $\mathcal{M}_i, x \vDash \Box_A\Box^*_{\mathbb{B}}\chi$. By Lemma 5.29, for all z for which $x \xrightarrow{A}_{i+1} z$, it is the case that $\mathcal{M}_i, x \vDash \Box^*_{\mathbb{B}}\chi$. By the truth lemma for $\mathcal{M}_i$ and consistency, we have that $\mathcal{M}_i, x, \vDash \chi$. Using a simple induction on path length, we establish for all z for which $x \xrightarrow{\mathbb{B}}{}^*_{i+1} z$ that $\mathcal{M}_i, z \vDash \chi$, $\mathcal{M}_i, z \vDash \Box^*_{\mathbb{B}}\chi$, and $\mathcal{M}_i, z \vDash \Box_A\Box^*_{\mathbb{B}}\chi$. We can then apply the inner inductive hypothesis to obtain $\mathcal{M}_{i+1}, z \vDash \chi$ for each z such that $x \xrightarrow{\mathbb{B}}{}^*_{i+1} z$; and hence $\mathcal{M}_{i+1}, x \vDash \Box^*_{\mathbb{B}}\chi$. The converse is more straightforward; rather than applying Lemma 5.29, the definition of the semantics of $\Box^*_{\mathbb{B}}$ can be used directly, as $\xrightarrow{A}_i$ is a subset of $\xrightarrow{A}_{i+1}$ and every epistemic path in $\mathcal{M}_i$ is an epistemic path in $\mathcal{M}_{i+1}$.

Case $\alpha\Box^*_{\mathbb{B}}\chi$: Suppose that $\psi = \alpha\Box^*_{\mathbb{B}}\chi$, $\mathsf{leafd}(x) = j$, and $\mathcal{M}_i, x \vDash \alpha\Box^*_{\mathbb{B}}\chi$. As $\mathcal{M}_i$ satisfies the truth lemma, we have by consistency that $\mathcal{M}_i, x \vDash t(\alpha\chi)$. Similarly, as the definition of cl guarantees that for each $A \in \mathbb{B}$ and β where $\alpha \xrightarrow{A} \beta$, $\Box_A\beta\Box^*_{\mathbb{B}}\chi \in cl_{\mathsf{leafd}(x)}(\varphi)$, we can apply Proposition 3.15 so that for each such β, if $\mathcal{M}_i, x \vDash t(\textsc{Pre}(\alpha))$, then $\mathcal{M}_i, x \vDash \Box_A\beta\Box^*_{\mathbb{B}}\chi$. Then assuming that $\mathcal{M}_i, x \vDash t(\textsc{Pre}(\alpha))$, we can apply Lemma 5.29 so see that for every z for which $x \xrightarrow{A}_{i+1} z$ that $\mathcal{M}_i, z \vDash \beta\Box^*_{\mathbb{B}}\chi$.

Using this reasoning, we can show by induction that for each pair of paths

$$x = x_0 \xrightarrow{A_1}_{i+1} x_1 \xrightarrow{A_2}_{i+1} \cdots \xrightarrow{A_{m-1}}_{i+1} x_{m-1} \xrightarrow{A_m}_{i+1} x_m,$$

and

$$\alpha_0 \xrightarrow{A_1} \alpha_2 \xrightarrow{A_2} \cdots \xrightarrow{A_{m-1}} \alpha_{m-1} \xrightarrow{A_m} \alpha_m,$$

where $\mathcal{M}_i, x_k \vDash t(\textsc{Pre}(\alpha_k))$ for $k \le m$, it is the case that $\mathcal{M}_i, x_m \vDash t(\alpha_m\chi)$. By the inner inductive hypothesis, $\mathcal{M}_{i+1}, x_m \vDash t(\alpha_m\chi)$, and hence by the definition of the semantics of ETMs, $\mathcal{M}_{i+1}, x \vDash \alpha\Box^*_{\mathbb{B}}\chi$.

The converse is more straightforward; rather than applying Proposition 3.15 and Lemma 5.29, the definition of the semantics of $\alpha\Box^*_{\mathbb{B}}$ in both $\mathcal{M}_i$ and $\mathcal{M}_{i+1}$ can be used directly, as $\xrightarrow{A}_i$ is a subset of $\xrightarrow{A}_{i+1}$ and every epistemic path in $\mathcal{M}_i$ is an epistemic path in $\mathcal{M}_{i+1}$. With paths using the same notation as above, we apply the inner induction hypothesis to establish $\mathcal{M}_{i+1}, x_k \vDash t(\textsc{Pre}(\alpha_k))$ from $\mathcal{M}_i, x_k \vDash t(\textsc{Pre}(\alpha_k))$ and then $\mathcal{M}_i, x_m \vDash t(\alpha_m\chi)$ from $\mathcal{M}_{i+1}, x_m \vDash t(\alpha_m\chi)$. $\Box$

Properties of $\mathcal{M}_H$

Lemma 5.31. *The model $\mathcal{M}_H$ is a characterizable epistemic temporal history.*

Proof. Properties 1 (*event points*), 2 (*partial functionality*), 3 (*bounded age*), and 5 (*states a*), do not involve the epistemic relations or the valuation, and are hence preserved from $\mathcal{M}_T$.

4. Synchronicity: This is established by an induction argument on the index i of the model. Note that $\mathcal{M}_0 = \mathcal{M}_T$, for which synchronicity holds. For every x and z for which $x \xrightarrow{A}_{i+1} z$, but where it is not the case that $x \xrightarrow{A}_i z$, there are e and f such that $(x, e) \xrightarrow{A}_i (z, f)$. By the inductive hypothesis, (x, e) and (z, f) have the same number of Y-relational steps to a Y-terminal state, and hence x and z will too.
6. States b: Given $e \in E$ and a number k for which $1 \leq k \leq n = \mathsf{dep}(\varphi)$, the formula q_e^k as defined in the definition of $\mathcal{M}_H$ will satisfy the requirement in the definition of this condition.
7. Relation a: Suppose $(x, e) \xrightarrow{A}_H (z, f)$. If $i = \mathsf{rootd}((x, e))$, then $(x, e) \xrightarrow{A}_i (z, f)$, and by definition of $\xrightarrow{A}_{i+1}$, we have that $x \xrightarrow{A}_{i+1} z$, whence $x \xrightarrow{A}_H z$.
8. Relation b: Suppose that $x \xrightarrow{A}_H z$, and $\varepsilon_H(x) = e \neq \emptyset$ and $\varepsilon(z) = f$. Then $\mathsf{leafd}(x) \geq 1$, and hence $e, \Box_A \neg f \in cl_{\mathsf{leafd}(x)}(\varphi)$. If it were not the case that $e \xrightarrow{A} f$, then by the truth lemma for $\mathcal{M}_H$, we would have that $\mathcal{M}_H, x \Box_A \neg f$, contradicting the fact that $x \xrightarrow{A}_H f$.
9. Relation c: Assume that $(x, e), (z, f) \in S_H$ and $x \xrightarrow{A}_H z$. Our notation indicates that $(x, e)Yx$ and $(z, f)Yz$. Since we have by hypothesis that $x \xrightarrow{A}_H z$, we have two distinct states related (via Y_H and $\xrightarrow{A}_H$) to a single state. Note that because of the unraveling, no two states are related in $\mathbf{S}_T$ to any one state, when one of the relational connections is Y. But the construction of $\mathbf{S}_H$ introduces some exceptions, all of a particular kind. All such exceptions arise from the introduction of arrows $x \xrightarrow{A}_H z$. Such an arrow is added when there exist (x, e') and (z, f') such that $(x, e') \xrightarrow{A}_H (z, f')$. In our case, if $e' \neq e$, then we would have distinct states (x, e) and (x, e') Y-related (in $\mathbf{S}_U$, $\mathbf{S}_T$, and $\mathbf{S}_H$) to the single state x, a contradiction. Hence $e' = e$, and by a similar argument, $f' = f$. We thus conclude that $(x, e) \xrightarrow{A}_H (z, f)$.
10. Valuation This condition is satisfied when the model obeys atomic permanence. Most of the proposition letters are untouched from how they were defined in $\mathcal{M}_T$, and hence inherit atomic permanence from $\mathcal{M}_T$. At issue are the proposition letters of the form q_e^k, and atomic permanence is guaranteed by the definition of Q_e^k. □

Completeness

Theorem 5.32 (*Weak Completeness*). *Given $\varphi \in \mathcal{L}_{e+Y}$, there is a sequential history $\mathcal{H}$ and a state $s \in \mathcal{H}$, such that $s \in [\![\varphi]\!](\mathcal{H})$.*

Proof. Given a formula $\varphi \in \mathcal{L}_{e+Y}$ that is consistent, there is a formula $\varphi_t \in \mathcal{L}_{\mathsf{rf}}$ such that $\varphi \equiv \varphi_t$, and hence φ_t is consistent. Since φ_t is a consistent formula in $\mathcal{L}_{\mathsf{rf}}$, we can create a model $\mathcal{M}_H$ as done in this section for which φ_t is satisfied. This model is, by Lemma 5.31, a characterizable epistemic temporal history. We can apply Lemmas 4.17 and 4.18 to get a characterizable sequential history $\mathcal{H}$ that satisfies φ_t. Since φ_t is satisfied in $\mathcal{H}$ and $\varphi_t \equiv \varphi$, we have that φ is also satisfied in $\mathcal{H}$. □

Decidability

Theorem 5.33 (*Decidability*). *Given $\varphi \in \mathcal{L}_{e+Y}$, it is decidable whether or not it is provable in $\mathcal{L}_{e+Y}$.*

Proof. Given a function φ, translating it into a formula in $\mathcal{L}_{\mathsf{rf}}$ is computable by Proposition 4.8. The closure function cl is also computable by Lemma 5.4, and it generates a finite set. Note that the history constructed in the proof of the completeness theorem is finite. An upper bound to the size of the filtration can be determined from φ. Suppose this upper bound is b and suppose n is the yesterday depth of φ. Then the size of the history will be bounded by nb^n. □

6. Extending the completeness proof

In this section we consider two proof systems, and their completeness and decidability. Recall that given an event frame $\mathcal{F}$ and a set Φ of atomic propositions, the languages $\mathcal{L}_{e+Y}(\mathcal{F}, \Phi)$ and $\mathcal{L}_Y(\mathcal{F}, \Phi)$ were defined in Section 2.3. Now let us define another language $\mathcal{L}_N(\mathcal{F}, \Phi)$ to be the language obtained by removing from $\mathcal{L}_{e+Y}(\mathcal{F}, \Phi)$ all formulas that have subformulas of the form $\Box_{\mathbb{B}}^* \varphi$.

Let $\mathcal{PS}_{e+Y}(\mathcal{F}, \Phi)$ be the proof system for $\mathcal{L}_{e+Y}(\mathcal{F}, \Phi)$, which was defined in Section 3.1 and for which completeness has been proved. Let $\mathcal{PS}_N(\mathcal{F}, \Phi)$ and $\mathcal{PS}_Y(\mathcal{F}, \Phi)$ be the proof systems obtained from $\mathcal{PS}_{e+Y}(\mathcal{F}, \Phi)$ by removing any axiom or rule that involves formulas not in respectively $\mathcal{L}_N(\mathcal{F}, \Phi)$ and $\mathcal{L}_Y(\mathcal{F}, \Phi)$.

6.1. Without common knowledge

We first consider the proof system $\mathcal{PS}_N(\mathcal{F}, \Phi)$. A proof of completeness and decidability of $\mathcal{PS}(\mathcal{F}, \Phi)$ is a simplification of the proof for $\mathcal{PS}_{e+Y}(\mathcal{F}, \Phi)$. First the reduced form language $\mathcal{L}_{\mathsf{rf}}$ will not have any occurrences of event modalities, and this will reduce many steps in inductive proofs. The filtration still need not have all the properties of an epistemic temporal history, but the unraveling can be simplified, so that the model is unraveled for all relations rather than just the relation Y. We can then trim the model according to the epistemic depth of the given consistent formula, as well as the yesterday depth, and this will ensure a finite model. The only step needed in $\mathcal{PS}_N$ not used in the proof for the completeness of $\mathcal{PS}_{e+Y}$ is the trimming of the unraveled canonical model at the epistemic depth of the given formula. Thus there is a proof of the completeness of $\mathcal{PS}_N$ that is cleaner and shorter, but still primarily the same as the proof of the completeness of $\mathcal{PS}_{e+Y}$.

6.2. Without expressing event points

Completeness and decidability of $\mathcal{PS}_Y$ is discussed in [16]. That paper stated that the $\mathcal{PS}_Y(\mathcal{F}, \Phi)$ is complete if $\mathcal{F}$ contains a point that is reflexive for every agent. Here we reflect on the proof of the completeness of $\mathcal{PS}_{e+Y}(\mathcal{F}, \Phi)$. Although the language $\mathcal{L}_{e+Y}$ does not involve event points, we may still use the exact same epistemic temporal models. The definition of the event point assignment function ε will not affect the truth of the formulas (as there there will be no event points in the language), but an appropriate definition will be important in establishing the 10 properties of an epistemic temporal history (Definition 4.14). So we may define ε in the filtration however we want, and redefine ε in any arbitrary way at each step until the last. If e is an event point that is reflexive for every agent, we make sure that $\varepsilon_H(x) = \emptyset$ if x is not Y_H related to any other state, and $\varepsilon_H(x) = e$ otherwise. The event point condition becomes immediately satisfied. The fact that e was reflexive for every agent means that the other properties (the ones regarding the update product relation) hold too.

It is in general unknown whether $\mathcal{PS}_Y(\mathcal{F}, \Phi)$ is complete when $\mathcal{F}$ does not have an event point reflexive for every agent. But there is an event frame $\mathcal{F}$ for which $\mathcal{PS}_Y(\mathcal{F}, \Phi)$ is incomplete, given by the following example found in [16]. Let $\mathcal{F}_1$ consist of exactly one event point e and empty epistemic relations, and let $\mathcal{F}_2$ consist of exactly one event point e, and let the e be reflexive for every agent. We know that $\mathcal{PS}_Y(\mathcal{F}_1, \Phi) = \mathcal{PS}_Y(\mathcal{F}_2, \Phi)$. As the proof system is complete and the formula $\Box_A\,\mathsf{false}$ is not valid when the event frame is $\mathcal{F}_2$, $\Box_A\,\mathsf{false}$ is not provable. But $\widehat{Y}\,\mathsf{true} \to \Box_A\,\mathsf{false}$ is valid when the event frame is $\mathcal{F}_1$.

The axiom schema in $\mathcal{PS}_{e+Y}(\mathcal{F}, \Phi)$ missing from $\mathcal{PS}_Y(\mathcal{F}, \Phi)$ include a few that correspond to at most one event point being associated with any state, another that relates the event point that is true to the event point that led to the state, and finally one that restricts the epistemic relation based on what event points are true. It is this last axiom scheme that captures properties that may be needed in a complete proof system for $\mathcal{L}_Y(\mathcal{F}, \Phi)$.

7. Conclusion

This paper proves completeness and decidability of a language that adds a discrete previous-time operator and formulas representing partial previous-events to a version of dynamic epistemic logic that includes common knowledge operators. One aspect of this paper that distinguishes it from other papers that add temporal operators to dynamic epistemic logic is that this paper involves common knowledge operators. On the one hand, without the common knowledge operator, we would be able to use the reduction technique of [9] to translate our language to one that does not involve any event modalities. Along similar lines, replacing the common knowledge operators with the relativized common knowledge operators of [4] might result in a proof more similar to one without any common knowledge operator, as a full reduction can be used to convert the language to one that does not include any event modalities as is done in [4,9], though this is left for future work. But on the other hand, the inclusion of the common knowledge operator in this paper adapts elegant techniques used in [3] to a broader setting. For example, as the reduction technique cannot completely eliminate all event modalities in the reduction process when common knowledge is used, the event rule, adapted from one in [3], gives us a way of dealing with this interaction between common knowledge and event modalities by justifying our choice of static semantics over ETMs for event modalities.

The completeness proof of this paper extends existing techniques, such as language reduction, filtration, and model unraveling, and integrates them with further model transformations in a novel way. Characteristic of this completeness proof is the shaping of an epistemic temporal model generated by a filtration to obey merging framework style properties in the sense of [5,6]. This paper offers many techniques that may inspire new developments in completeness proofs in DEL or other related logics.

Acknowledgements

I am grateful for the help that I have received from Lawrence Moss. I also thank Alexandru Baltag, Johan van Benthem, Esfandiar Haghverdi, Chung-min Lee, Daniel Leivant, and Eric Pacuit for their comments.

The author was partly supported by the projects "New Developments in Operational Semantics" (nr. 080039021) and "Processes and Modal Logics" (nr. 100048021) of The Icelandic Research Fund and a grant from Reykjavík University's Development Fund.

References

[1] P. Balbiani, A. Baltag, H. van Ditmarsch, A. Herzig, T. Hoshi, T. de Lima, Knowable as known after an announcement, in: The Review of Symbolic Logic, 1, Cambridge University Press, 2008, pp. 305–334.
[2] A. Baltag, L.S. Moss, Logics for epistemic programs, Synthese 139 (2) (2004) 165–224. Knowledge, Rationality & Action.
[3] A. Baltag, L. Moss, S. Solecki, Logics for Epistemic Actions: Completeness, Decidability, Expressivity. ms. Indiana University, 2003.
[4] J. van Benthem, J. van Eijck, B. Kooi, Logics of communication and change, Information and Computation 204 (11) (2006) 1620–1662.
[5] J. van Benthem, J. Gerbrandy, T. Hoshi, E. Pacuit, Merging frameworks for interaction, Journal of Philosophical Logic 38 (5) (2009) 491–526.
[6] J. van Benthem, J. Gerbrandy, E. Pacuit, Merging frameworks for interaction: DEL and ETL, in: Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge, TARK XI, 2007, pp. 72–81.
[7] J. van Benthem, F. Liu, Diversity of logical agents in games, Philosophica Scientiae 8 (2) (2004) 163–178.
[8] J. van Benthem, E. Pacuit, The tree of knowledge in action: towards a common perspective, in: Proceedings of Advances in Modal Logic, 2006.

[9] H. van Ditmarsch, W. van der Hoek, B. Kooi, Dynamic Epistemic Logic, Springer, 2008.
[10] R. Fagin, J. Halpern, Y. Moses, M. Vardi, Reasoning About Knowledge, The MIT Press, Boston, 1995.
[11] D. Gerbrandy, Bisimulations on Planet Kripke. Dissertation, ILLC, 1998.
[12] H. Hoshi, A. Yap, Dynamic epistemic logic with branching temporal structures, Synthese 169 (2) (2009) 259–281.
[13] R. Parikh, R. Ramanujam, A knowledge based semantics of messages, Journal of Logic, Language, and Information 12 (2003) 453–467.
[14] J. Plaza, Logics of public communications, in: proceedings, 4th International Symposium on Methodologies for Intelligent Systems, 1989.
[15] B. Renne, J. Sack, A. Yap, Dynamic epistemic temporal logic, in: Proceedings of Second International Workshop in Logic, Rationality and Interaction, 2009.
[16] J. Sack, Temporal languages for epistemic programs, Journal of Logic, Language and Information 17 (2) (2008) 183–216.
[17] J. Sack, Extending probabilistic dynamic epistemic logic, Synthese 169 (2) (2009) 241–257.